On the Efficiency of Networked Stackelberg Competition

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***Abstract*—We study the impact of strategic anticipative behavior in networked markets. We focus on the case of electricity markets and model the market as a game between a system operator (market maker) and generators at different nodes of the network. Generators submit quantity bids and the system operator balances demand and supply over the network subject to transmission constraints. We compare the efficiency of a networked Stackelberg equilibrium, where generators anticipate the market clearing actions of the market maker, with a networked Cournot equilibrium, where they do not. We show that networked Cournot equilibria always exists but its efficiency loss is unbounded in the worst case. In contrast, networked Stackelberg equilibria do not always exist, but in certain settings where they do exist, the efficiency loss may be bounded above by a constant.**

I. Introduction

Classical oligopoly models of competition have focused on a single marketplace, where the identity of participants has no impact on prices, outcomes, etc. However, marketplaces today are typically more complex, and often, the options available to individual participants are varied and highly constrained. That is, the marketplace is not really a single market anymore, but is instead more naturally characterized as a network of interconnected markets with constraints on the graph of feasible exchanges.

Recently, the study of such "networked marketplaces" has garnered significant attention across economics, electrical engineering, and computer science, motivated by the importance of such models in application areas like electricity markets, financial markets, and the study of intermediaries, exchanges, etc. Naturally, a variety of models have surfaced that capture different forms of competition in networked marketplaces. The two most prominent such models are networked Bertrand competition, e.g., [1]–[3], and networked Cournot competition, e.g., [4]–[7].

Our work is motivated by wholesale electricity markets. The interactions in an electricity markets are typically complex and generally challenging to analyze. The networked Cournot competition model has proved to be simple yet sufficiently rich to analyze market power issues in such markets. It is particularly attractive in that it allows one to incorporate a linearized power flow model that constrains the allocation of generation to demand, while remaining amenable to mathematical analysis. For a non-exhaustive list of the literature on networked Cournot framework for electricity market analysis, see [6]–[13].

A. Contributions of this paper

Our focus in this paper is on a variant of the networked Cournot competition – the *networked Stackelberg competition*. To motivate the same, consider a networked marketplace with a market maker who facilitates the balance of supply and demand within the confines of the underlying network. The Independent System Operators (ISO) play the role of such a market maker in the various wholesale electricity markets [14], [15]. Also, exchanges such as NYSE, Yahoo Ad Exchange, NGX, CME, etc., are managed by a market maker. In such a centrally managed networked marketplace, it is natural for players to *anticipate* the outcome of the market clearing rule that the market maker implements. This anticipation leads one away from Bertrand or Cournot competition towards a Stackelberg competition, where the participants are the Stackelberg leaders and the market maker is the follower.

Our main results seek to understand the impact of the aforementioned anticipatory behavior in networked marketplaces. More precisely, we focus on a contrast between networked Cournot competition (non-anticipatory behavior) and networked Stackelberg competition (anticipatory behavior) and derive comparisons with respect to both the existence and the efficiency of equilibria. We show that networked Cournot competition has an unbounded price of anarchy[1] even in the case of two-node unconstrained networks (Proposition 1), while networked Stackelberg has a price of anarchy that is bounded by a small constant in many cases (see Theorems 1 and 2). However, an equilibrium always exists in the network Cournot model for the case of general constrained networks, whereas an equilibrium need not exist in the networked Stackelberg model, even if the network is unconstrained. In short, anticipatory behavior leads to more efficient market outcomes, albeit risking a possible market failure owing to a lack of equilibrium.

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[1]The price of anarchy is defined as the worst-case ratio of an equilibrium welfare to the social optimal welfare.

B. Related Literature

Our focus in this work is on the impact of anticipatory behavior in networked marketplaces that are managed by a market maker. To the best of our knowledge, the networked Stackelberg model we consider is novel. The networked Cournot model we use for comparison, however, has a long history, both in the context of electricity markets and beyond.

Classical Cournot competition (without a network) dates back to the nineteenth century to the work of Cournot himself [16]. This competition model and its analysis can be found in standard texts in microeconomics, e.g., [17]. Of recent interest has been the study of efficiency loss due to strategic behavior in this framework [18]–[20]. These works derive bounds on the efficiency loss under a variety of model assumptions. We leverage such results in our analysis of networked Stackelberg competition.

Cournot competition over a networked market has been studied in the context of electricity markets in [9], [11]. Beyond electricity markets, such models have been analyzed in [4], [5]. Competition models à la Cournot for the electricity market come in two general flavors: (i) prices due to transmission congestion over the network are exogenously determined by the market maker, and (ii) prices are determined endogenously through a nodal inverse demand function. Early works by Jing-Yuan et al. [8], Oren [9], and Willems [10] belong to the first category. Papers by Metzler et al. [11], and others [12], [13] adopt the second approach. See [11] for a nuanced discussion of various Cournot competition models in electricity markets. Closest in spirit to our work is [6], [7], which study the role of a market maker in a networked Cournot competition.

II. Model and preliminaries

We model networked Cournot and Stackelberg competitions as strategic games in this section. These models will allow us to compare the effect of anticipatory behavior of the market participants on the market outcomes. Recall that electricity markets serve as our motivating application, and hence, our formulation will incorporate network constraints defined by Kirchhoff's laws.

The following notation will prove useful in the sequel. Let $\mathbb{R}$ (resp. $\mathbb{R}_+$) denote the set of real (resp. nonnegative real) numbers. We distinguish vectors and matrices via boldface symbols. For any two vectors $\mathbf{u}$ and $\mathbf{v}$ of the same dimension, we say $\mathbf{u} \geq \mathbf{v}$, if the inequality holds elementwise. Let $\mathbf{1}$ denote the vector of all ones of appropriate size. For any vector $\mathbf{v}$, let $\mathbf{v}^\top$ denote its transpose, and $\mathbf{v}_{-i}$ denote the vector of all elements in $\mathbf{v}$ except the i-th element.

A. Network model

Consider a power network on n buses (or nodes), labelled $1, \ldots, n$, and ℓ transmission lines (or edges). The power flows over the transmission lines are related to the nodal power injections through Kirchhoff's laws. Together with constraints on the transmission line flows, the true power flow model defines a nonconvex set of feasible nodal power injections. However, electricity market operations typically rely on a linearized power flow model that utilizes the so-called *DC approximations* [14]. See [21], [22] for more details on these approximations that allow us to succinctly represent the set of nodal power injections as

$$\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{H}\mathbf{x} \leq \mathbf{f},\ \mathbf{1}^\top \mathbf{x} = 0\}, \tag{1}$$

where $\mathbf{H} \in \mathbb{R}^{2\ell \times n}$ is the so-called *shift-factor matrix* that maps the nodal power injections to (directional) transmission line flows, and $\mathbf{f} \in \mathbb{R}_+^{2\ell}$ denotes the transmission line capacities.

B. Market participants

An electricity market is comprised of the consumers (represented in aggregate by the load serving entities), the generators, and the market maker (usually an ISO/RTO). In what follows, we model the consumers as price-takers, and then formulate the spot market clearing as a networked Stackleberg and Cournot competitions between the generators and the market maker. Generators are modeled as strategic agents, while the market maker plays the role of a social planner that seeks to maximize social welfare, while maintaining demand/supply balance within the confines of the power network.

Consumers: Let $p_k : \mathbb{R}_+ \to \mathbb{R}$ denote the inverse demand function of the (aggregate) consumer at node k. In other words, the consumer is willing to pay $p_k(d_k)$ to consume d_k units of power. Assume p_k to be twice continuously differentiable, strictly decreasing, and concave, i.e., $p_k' < 0$ and $p_k'' \leq 0$.

Generators: Assume that each node k has a generator G_k that submits a quantity offer $q_k \geq 0$. It incurs a cost of $c_k(q_k)$ for producing q_k. Suppose $c_k : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable, strictly increasing, and convex, with $c_k(0) = 0$. For convenience, define

$$\mathbf{q} := (q_1, \ldots, q_n)^\top.$$

Market maker: The market maker employs a market mechanism to choose a vector of 'rebalancing quantities', denoted by $\mathbf{r} := (r_1, \ldots, r_n)^\top \in \mathbb{R}^n$. Here, $-r_k$ denotes the net power injection into node k. The choice of $\mathbf{r}$ allocates the nodal demands d_k via the power balance at node k, given by

$$d_k := q_k + r_k.$$

In turn, it sets the nodal price to $p_k(d_k) = p_k(q_k + r_k)$.

C. Payoff functions

Recall that generator G_k chooses q_k, while the market maker chooses $\mathbf{r}$. The generator is paid at a rate given by the local nodal price $p_k(q_k + r_k)$. This in turn defines the profit (or the payoff) of generator G_k as

$$\pi_k(\mathbf{q}, \mathbf{r}) := q_k p_k(q_k + r_k) - c_k(q_k), \tag{2}$$

that she aims to maximize over $q_k \in \mathbb{R}_+$.[2] Following prior literature [6]–[8], [12], we assume that the market maker is

[2]The generators are assumed to have infinite supply capacities. Supply constraints will no doubt affect the equilibrium behavior. The unconstrained case, however, suffices to reveal the impact of anticipatory behavior of market participants – the object of study of the current work. In addition, we remark that such assumptions are common in the literature (cf. [5], [20]).

a social planner who seeks to maximize the social welfare, defined as

$$\Pi(\mathbf{q}, \mathbf{r}) := \sum_{k=1}^{n} \left(\int_0^{q_k + r_k} p_k(w_k) dw_k - c_k(q_k) \right), \qquad (3)$$

over $\mathbf{r} \in \mathcal{X}$.

D. Competition models

Equipped with the above notation, we now define two different models of competition.

1) *Cournot competition*: Generators $G_1, \ldots, G_n$, and the market maker move simultaneously.
2) *Stackelberg competition*: The generators participate in a simultaneous game at the first stage. The market-maker moves second.

The two formulations differ in terms of the anticipatory behavior on the part of the generators. More precisely, each generator accounts for the response of the market maker in its quantity offer in the Stackelberg game. Non-anticipative generators in the Stackelberg formulation will reduce the game to a Cournot one.

Contrary to electricity markets – where game parameters such as generator costs and line flow limits may not be common knowledge – our competition models are presented as games of *complete information*. Albeit idealized, these models are rich enough to reveal the impact of anticipatory behavior of generators in such networked marketplaces. We call the games *feasible*, if $\mathcal{X}$ is nonempty.

We next define the equilibrium concepts for the two competition models. We say $(\mathbf{q}^{\mathrm{C}}, \mathbf{r}^{\mathrm{C}})$ constitutes a Cournot equilibrium, if

$$\pi_k\left(q_k^{\mathrm{C}}, \mathbf{q}_{-k}^{\mathrm{C}}, \mathbf{r}^{\mathrm{C}}\right) \geq \pi_k\left(q_k, \mathbf{q}_{-k}^{\mathrm{C}}, \mathbf{r}^{\mathrm{C}}\right),$$
$$\Pi\left(\mathbf{q}^{\mathrm{C}}, \mathbf{r}^{\mathrm{C}}\right) \geq \Pi\left(\mathbf{q}^{\mathrm{C}}, \mathbf{r}\right),$$

for all $q_k \in \mathbb{R}_+$ and $\mathbf{r} \in \mathcal{X}$, where $k = 1, \ldots, n$.

To define a Stackelberg equilibrium, we need additional notation. Denote by $\rho : \mathbb{R}_+^n \to \mathcal{X}$, the reaction function of the market maker that she utilizes to map the vector of offered quantities $\mathbf{q} \in \mathbb{R}_+^n$ to a vector of rebalancing quantities $\mathbf{r}$. Then, $\left(\mathbf{q}^{\mathrm{S}}, \rho^{\mathrm{S}}\right)$ constitutes a Stackelberg equilibrium, if

$$\pi_k\left(q_k^{\mathrm{S}}, \mathbf{q}_{-k}^{\mathrm{S}}, \rho^{\mathrm{S}}\left(q_k^{\mathrm{S}}, \mathbf{q}_{-k}^{\mathrm{S}}\right)\right) \geq \pi_k\left(q_k, \mathbf{q}_{-k}^{\mathrm{S}}, \rho^{\mathrm{S}}\left(q_k, \mathbf{q}_{-k}^{\mathrm{S}}\right)\right),$$

for all $q_k \in \mathbb{R}_+$, where $k = 1, \ldots, n$, and

$$\Pi(\mathbf{q}, \rho^{\mathrm{S}}(\mathbf{q})) \geq \Pi(\mathbf{q}, \rho(\mathbf{q})),$$

for all $\rho : \mathbb{R}_+^n \to \mathcal{X}$ and $\mathbf{q} \in \mathbb{R}_+^n$.

III. Summary of results

Our focus is on understanding the impact of anticipatory behavior in the networked competition models formulated above. We thus contrast the equilibrium outcomes of the networked Cournot and the networked Stackelberg models, both in terms of the existence and the efficiency of the equilibria. We will state our results on efficiency in terms of the *price of anarchy*, denoted by PoA, which equals the worst-case ratio of the social welfare at an equilibrium outcome to the optimal achievable social welfare.

A. Existence and efficiency of Cournot equilibria

The following result characterizes the existence and efficiency of equilibria in networked Cournot competition.

Proposition 1. *Suppose the Cournot game is feasible.*

1) *An equilibrium always exists.*
2) *The price of anarchy can be arbitrarily large, even in an unconstrained two-node network with affine inverse demand functions and quadratic cost functions.*

The existence of equilibria was reported in [6], [7]; however the price of anarchy result is novel to this work (see Section IV). It is perhaps surprising how bad the efficiency can be, given that the market maker is attempting to clear the market by maximizing the social welfare. The unbounded price of anarchy can be shown using the simplest non-trivial setting of two-node networks with affine inverse demand functions.

B. Existence and efficiency of Stackelberg equilibria

Since Stackelberg competition considers anticipatory behavior *on top of* the networked Cournot model, which is already difficult to analyze, one can expect that it is challenging to obtain general results characterizing networked Stackelberg equilibria. Further, one might expect equilibria to both exist less often and be more inefficient (have a larger price of anarchy) than the networked Cournot model. Interestingly, however, there is a large regime where networked Stackelberg is actually simpler to analyze and more efficient than the networked Cournot model.

Perhaps surprisingly, if the network is unconstrained, it turns out that a general class of networked Stackelberg models (when the price intercepts of the inverse demand function are spatially homogenous) can be reduced to a classical (non-networked) Cournot competition (see Section V). Known results on that (simpler) model then yield the existence and price of anarchy results. In contrast, such a reduction is not possible for an unconstrained networked Cournot model. As a result, the anticipatory nature of the networked Stackelberg model actually simplifies the analysis. Moreover, in such settings, the networked Stackelberg model has a smaller price of anarchy than the networked Cournot model – 3/2 instead of unbounded. In essence, the anticipatory behavior makes the marketplace more efficient!

It is, however, important to point out that, outside of the settings where such a reduction is possible, networked Stackelberg competition may not admit an equilibrium.

Theorem 1 (Unconstrained network)**.** *Suppose the Stackelberg game is feasible with* $\mathbf{f} = \infty$.

1) *An equilibrium does not always exist. An equilibrium, however, always exists if the price intercepts of the inverse demand functions are spatially homogeneous.*[3]
2) *The price of anarchy is bounded above by* $3/2$*, when the inverse demand functions are affine with spatially homogenous price intercepts.*

[3]That is, $p_k(0)$'s are identical for all $k = 1, \ldots, n$.

While the setting where we attain positive results may seem limited, note that the class of affine inverse demand functions is commonly studied in economics [23] and in electricity market literature [12], [13].

Not surprisingly, when network constraints are included, the characterization of networked Stackelberg equilibria becomes much more difficult. In such situations, it is challenging to provide general results about either existence or efficiency. In the case of existence, we can construct examples where no equilibrium exists even in the setting where the price intercepts of the inverse demand functions are spatially homogenous. In the case of efficiency, obtaining bounds in two-node networks is already difficult. However, as the following theorem highlights, two-node networks, where the price intercepts of the inverse demand functions are spatially homogenous, have a similarly strong bound on the price of anarchy. This is reminiscent of the case with unconstrained networks.

Theorem 2 (Constrained network)**.** *Suppose the Stackelberg game is feasible with a finite* $\mathbf{f}$.

1) *An equilibrium does not always exist, even if the price intercepts of the inverse demand functions are spatially homogeneous.*
2) *The price of anarchy for two-node networks is bounded above by* $4/3$*, when the inverse demand functions are affine with spatially homogenous price intercepts and costs are quadratic.*

Note that there is an additional assumption on the cost functions in the above price of anarchy result as compared with Theorem 1. Again, this may seem limiting; quadratic cost functions, however, have been widely adopted in recent works that apply Cournot models to study electricity markets [6], [10], [11].

Further, while it may seem strange that Theorem 2 has a tighter bound on the price of anarchy than Theorem 1, note that when the line capacity is unconstrained, the same upper bound of $4/3$ can be derived for two-node networks with quadratic costs. This highlights the possibly surprising fact that adding network constraints does not reduce efficiency in this context.

Theorem 2 is far from a complete characterization of equilibria in the constrained network setting. However, it represents a provocative start. An exact characterization of existence and a more general characterization of efficiency in the constrained network setting are certainly interesting and challenging directions for future work.

IV. The Inefficiency of Networked Cournot

In this section, we provide an example illustrating that the networked Cournot model has an unbounded price of anarchy, even in simple settings. Combined with the existence results in [6], [7], proves Proposition 1.

Consider a two-node network ($n = 2$) where the transmission line joining nodes 1 and 2 has an infinite capacity. Assume the following (parameterized) inverse demand and cost functions

$$\begin{aligned} p_1(d_1) = 1 - d_1, \;\; p_2(d_2) = 1 - \gamma d_2, \\ c_1(q_1) = \gamma q_1^2, \;\; c_2(q_2) = q_2^2, \end{aligned}$$

where $\gamma > 0$. With a slight abuse of notation, assume $r = r_1 = -r_2$. The social optimal allocation (q_1^*, q_2^*, r^*) then satisfies

$$\begin{aligned} q_1^*(1+2\gamma) + r^* &= 1, \\ q_2^*(2+\gamma) - \gamma r^* &= 1, \\ q_1^* - \gamma q_2^* + (1+\gamma) r^* &= 0. \end{aligned}$$

It follows from maximizing Π with respect to its arguments q_1, q_2, r. The solution to the above system of linear equations yields the following maximum attainable social welfare.

$$\Pi(q_1^*, q_2^*, r^*) = \frac{\gamma+1}{6\gamma}. \tag{4}$$

Next, consider a Cournot equilibrium defined by $\left(q_1^{\mathrm{C}}, q_2^{\mathrm{C}}, r^{\mathrm{C}}\right)$. Then, maximizing the payoff function of each player within her strategy set yields the following system of equations for $\left(q_1^{\mathrm{C}}, q_2^{\mathrm{C}}, r^{\mathrm{C}}\right)$.

$$\begin{aligned} q_1^{\mathrm{C}}(2+2\gamma) + r^{\mathrm{C}} &= 1, \\ q_2^{\mathrm{C}}(2+2\gamma) - \gamma r^{\mathrm{C}} &= 1, \\ q_1^{\mathrm{C}} - \gamma q_2^{\mathrm{C}} + (1+\gamma) r^{\mathrm{C}} &= 0. \end{aligned}$$

One can show that the solution of the above linear equations defines the unique equilibrium in the Cournot game, the social welfare at which is given by

$$\Pi(q_1^{\mathrm{C}}, q_2^{\mathrm{C}}, r^{\mathrm{C}}) = \frac{5\left(2\gamma^3+7\gamma^2+7\gamma+2\right)}{8\left(\gamma^2+4\gamma+1\right)^2}. \tag{5}$$

Relations (4) and (5) then yield

$$\lim_{\gamma\uparrow\infty} \frac{\Pi(q_1^*, q_2^*, r^*)}{\Pi(q_1^{\mathrm{C}}, q_2^{\mathrm{C}}, r^{\mathrm{C}})} = \infty.$$

Thus, the price of anarchy in the Cournot game is unbounded.

V. Stackelberg Competition in Unconstrained Networks

In this section we study the Stackelberg competition in unconstrained networks, and outline the proof of Theorem 1. Crucial to our analysis is the insight that, in the case where price intercepts are spatially homogenous, any equilibrium of our Stackelberg game is also an equilibrium of a classical non-networked Cournot game with an inverse martket demand curve aggregated from the individual nodal demand functions. This insight allows us to leverage existing results for classical Cournot games.

A. Existence

Let the common price intercept of the inverse demand functions be given by $p^0 > 0$. Recall that $p_k : \mathbb{R}_+ \to \mathbb{R}$ is assumed to be concave and monotonically decreasing. Therefore, $p_k^{-1} : (-\infty, p^0] \to \mathbb{R}_+$ is well-defined for each $k = 1, \ldots, n$. Using p_k^{-1}, define $D : (-\infty, p^0] \to \mathbb{R}_+$ as

$$D(x) := \sum_{k=1}^{n} p_k^{-1}(x). \tag{6}$$

We refer to D as the *market demand function* in the sequel. It is straightforward to verify that D is concave and strictly decreasing. It also admits an inverse $D^{-1} : [0, \infty) \to (-\infty, p^0]$. The following result establishes the relationship between the networked Stackelberg game and a classical Cournot game. Its formal proof is omitted due to space constraints.

Lemma 1. *Let $(\mathbf{q}^S, \rho^S)$ be an equilibrium of the networked Stackelberg game with $\mathbf{f} = \infty$ and inverse demand functions with equal price intercepts. Then $\mathbf{q}^S$ defines an equilibrium of a classical non-networked Cournot game between the generators with a market demand function D.*

Classical Cournot competition with a concave market demand function is known to admit an equilibrium (cf. [24]), implying the existence result in Theorem 1.

When the price intercepts of the inverse demand functions are not identical, the market demand function D may not be concave. In such cases, equilibrium may not exist. As an example, consider a two-node unconstrained network with affine inverse demand functions

$$p_1(d_1) = \gamma - d_1, \quad p_2(d_2) = 1 - d_2,$$

where $\gamma \in (9/5, 11/6)$, and quadratic costs $c_k(q_k) = q_k^2$ for $k = 1, 2$. With some effort, it can be shown that there does not exist a Stackelberg equilibrium in such a setting. The details are omitted due to space constraints.

B. Efficiency

We now move to characterizing the price of anarchy of Stackelberg equilibria in unconstrained networks. These results quickly follow from Lemma 1 that shows that any equilibrium of our Stackelberg game is also an equilibrium of a classical Cournot game with a concave market demand function (expressed in Eq. (6)). The socially optimal allocation in the Stackelberg game also happens to coincide with that in the classical Cournot one. Therefore, results in the literature on the PoA of classical Cournot games apply directly to our networked Stackelberg game. For instance, the result in [19] implies that the PoA of our Stackelberg game could be arbitrarily high under general concave demand functions. When the inverse demand functions at all nodes are affine with homogenous price intercepts – in which case D^{-1} is affine – the result in [18, Theorem 12] implies that PoA of our Stackelberg game has an upper bound of $3/2$.

VI. Stackelberg Competition in Constrained Networks

As mentioned in Section III, it is natural to expect Stackelberg equilibria to be much more difficult to characterize for constrained networks. This is indeed borne out in the generality of the results we obtain and the fact that the analysis needed to obtain the (less general) results is more involved.

A. Existence

Existence is far from guaranteed in this setting. In fact, even if the price intercepts of the demand functions are uniform across the network, an equilibrium may not exist if the network has a single line with a finite line capacity. To highlight this, again consider a two-node network, where the demand and cost functions at nodes $k = 1, 2$ are given by $p_k(d_k) = 1 - d_k$, and $c_k(q_k) = c_k q_k^2$, respectively. Assume $c_1 > c_2 > 0$. With some effort, we are able to show that if the transmission line joining buses 1 and 2 has a capacity of $f \in \mathbb{R}_+$ for power flowing in either direction, and f satisfies

$$f \geq \frac{c_1 - c_2}{4(c_1+1)(c_2+1) - (c_1+1) - (c_2+1)},$$
$$f \leq \frac{c_1 - c_2}{(1+2c_1)(1+2c_2) - 1/4},$$

an equilibrium does not exist in the Stackelberg game. The details are omitted due to space constraints.

B. Efficiency

In contrast with the negative result on existence, we can prove a positive result about efficiency, albeit in the limited setting of a two-node network. Recall that we have established an upper bound of $3/2$ in Theorem 1 for the price of anarchy of a Stackelberg game in unconstrained networks. To obtain Theorem 2, we require the following lemma which establishes a refined price of anarchy bound of $4/3$ for the special case of $n = 2$ with affine inverse demands and quadratic costs. The proof is omitted due to space constraints.

Lemma 2. *Consider a network with $n = 2$ buses, where the transmission line between the nodes have infinite capacity. If the inverse demand and cost functions have the form*

$$p_k(d_k) = a - b_k d_k, \quad c_k(q_k) = c_k q_k^2,$$

where $a, b_k, c_k > 0$, $k = 1, 2$, then the price of anarchy of the networked Stackelberg game is at most $4/3$.

With a slight abuse of notation, let $r = r_1 = -r_2$, and the market maker reacts to generators' actions through the map $\rho : \mathbb{R}^2 \to \mathbb{R}$ by choosing r. We are now ready to outline the major steps in the proof of the second part of Theorem 2. The details are omitted.

We seek to establish the $4/3$ bound in the setting of Lemma 2, but with finite line capacity. Let $f \in \mathbb{R}_+$ denote this line capacity in either direction. Define the map

$$h(\mathbf{q}) := \frac{b_2 q_2 - b_1 q_1}{b_1 + b_2}. \tag{7}$$

for $\mathbf{q} = (q_1, q_2)^\top \in \mathbb{R}_+^2$. Then, we have $-q_1 \leq h(q_1, q_2) \leq q_2$. Also, h admits the following interpretation. If the line capacity were infinite, $h(\mathbf{q})$ would define the market maker's best response to the generators' choice of $\mathbf{q}$.

Next, consider the equilibrium generator productions $\mathbf{q}^{\mathrm{S}} := (q_1^{\mathrm{S}}, q_2^{\mathrm{S}})^\top$. Then, $h(\mathbf{q}^{\mathrm{S}})$ satisfies one of the following three cases, that we tackle separately.

- $h(\mathbf{q}^{\mathrm{S}}) = f$. Using first-order conditions that characterize a Stackelberg equilibrium, one can show that an equilibrium with $h(\mathbf{q}^{\mathrm{S}}) = f$ does not exist.
- $h(\mathbf{q}^{\mathrm{S}}) > f$. [4] For an equilibrium with $h(\mathbf{q}^{\mathrm{S}}) > f$, it can be argued that $\rho^{\mathrm{S}}(\mathbf{q}^{\mathrm{S}}) = f$. Further, if $(\mathbf{q}^*, r^*)$ denotes the social optimal allocations, then it can be shown that $h(\mathbf{q}^*) > f$ and $r^* = f$. In essence, the congestion of the line in equilibrium implies a congestion in the socially optimal outcome. It can be further shown that the analysis of the price of anarchy of the entire market reduces to one of studying that in two separate markets, where the outcomes (in equilibrium and at social optimum) at each node depends only on the nodal demand and cost functions. The PoA bound then follows from [19, Theorem 3].
- $h(\mathbf{q}^{\mathrm{S}}) \in (-f, f)$. Within this case, if further $r^* < f$, the rest then follows from Lemma 2. Otherwise, the optimal social welfare at $(\mathbf{q}^*, r^* = f)$ is bounded above by the optimal social welfare obtained with $f = \infty$. Again, Lemma 2 provides the necessary bound.

VII. Conclusion and Future Work

We study the behavior of equilibria in a quantity competition among spatially distributed generators, where the market-maker maximizes social welfare subject to demand/supply balance, and flow constraints of the underlying power system. We study two different game formulations, based on whether the profit-maximizing generators do or do not anticipate the effect of the market-maker's actions in tailoring their quantity offers. Non-anticipatory generators compete in a simultaneous-move networked Cournot game, while the anticipatory case results in a networked Stackelberg formulation, where the generators move first, followed by the market-maker.

Our analysis reveals that the Cournot game admits an equilibrium under very general settings, but its efficiency loss (quantified through price of anarchy) can be arbitrarily bad. In contrast, the sequential formulation as a Stackelberg game, being a networked generalization of the classical Cournot competition, has limited efficiency loss (in the context of a two-node network), but an equilibrium may not exist outside a restrictive class of model parameters. Moving forward, a natural question to pursue is the generalization of the efficiency results in the Stackelberg competition to general networks. Another direction is to find an easily computable bound on the efficiency losses in both frameworks as a function of the network parameters and demand curves, rather than computing the worst case over the family of all network parameters.

[4]The analysis for $h(\mathbf{q}^{\mathrm{S}}) < -f$ is equivalent to the case $h(\mathbf{q}^{\mathrm{S}}) > f$, and is therefore omitted.

Analysis of such oligopoly models of spot electricity markets provide valuable insights into expected market outcomes and suggest possible amends to market design. The role of the market design in the Cournot game has been analyzed in [6], [7]. We hope to extend our analysis to the market design question in the networked Stackelberg setting.

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