On the Role of a Market Maker in Networked Cournot Competition

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Abstract

We study Cournot competition among firms in a networked marketplace that is centrally managed by a market maker. In particular, we study a situation in which a market maker facilitates trade between geographically separate markets via a constrained transport network. Our focus is on understanding the consequences of the design of the market maker and on providing tools for optimal design. To that end we provide a characterization of the equilibrium outcomes of the game between the firms and the market maker. Our results highlight that the equilibrium structure is impacted dramatically by the market maker objective – depending on the objective there may be a unique equilibrium, multiple equilibria, or no equilibria. Further, the game may be a potential game (as in the case of classical Cournot competition) or not. Beyond characterizing the equilibria of the game, we provide an approach for designing the market maker in order to optimize a design objective (e.g., social welfare) at the equilibrium of the game. Additionally, we use our results to explore the value of transport (trade) and the efficiency of the market maker (as compared to a single, aggregate market).

1 Introduction

The ubiquity of networks in our world today has had a fundamental impact on modern marketplaces. Classical models of competition often feature multiple firms operating in a single, isolated market; however power systems, the internet, transportation networks, infrastructure networks, and global supply chains are just a few of the places where varied and complex interconnections among participants are crucial to understanding and optimizing marketplaces. Consequently, the study of competition in networked markets has emerged as an area with both rich theoretical challenges and important practical applications.

At this point, a wide variety of models for competition in networked markets have emerged across economics, operations research, and computer science. The work in this literature focuses both on extensions of classical models of competition to networked settings, e.g., networked

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Bertrand competition [Guz11, AS15, CR08, ABO09] and networked Cournot competition [BEI14, ABH+14, Ilk09], and on models of specific applications where networked competition is fundamental, e.g., electricity markets [NBB+05, BV08, BV05, YOA04, YOA07, YAO08, JYS99].

Intermediaries, market makers, and transport

The complexity of networked marketplaces typically leads to (and often necessitates) the emergence of intermediaries. A prominent illustration of this is financial markets, where central core banks intermediate trade between smaller periphery banks. Similar examples are common in infrastructure networks: natural gas is traded through pipelines, which are managed by a Transmission System Operator (TSO), and transport in electricity markets is governed by an Independent System Operator (ISO). One can view platforms in the sharing economy, e.g., Uber, as intermediaries between service providers and customers, and supply chains can be regarded as a form of intermediation in networked markets.

Intermediaries can play many roles in networked markets, from aggregation to risk mitigation to informational and beyond. Our focus in this paper is on the role intermediaries play with respect to transport and trade. In particular, in many networked marketplaces participating firms depend on an intermediary, a.k.a., market maker, to provide transport of their goods between geographically distinct markets.

A particularly prominent example, which we use as the motivation throughout this paper, is *electricity markets*. In these markets, the ISO solves a centralized dispatch problem by utilizing the offers/bids from the generators/retailers. This problem seeks to maximize some metric of social benefit subject to the operational constraints of the grid. These operational constraints include physical laws that govern the flow of power in the network as well as safety constraints such as line capacity limits. The payments are calculated based on locational marginal prices (LMP). Therefore, the ISO plays a crucial role in matching the demand and supply of power within the confines of the grid and also define payments to the market participants. As an independent regulated entity, it further designs rules to limit the possible exercise of market power by the suppliers.

Beyond electricity markets, natural gas markets, and more generally, supply chains often have a similar structure where a market maker manages transport between geographically distributed markets.

Clearly, the design of the market maker in such situations is crucial to the efficiency of the marketplace. By facilitating trade, the market maker is providing a crucial opportunity for increased efficiency. However, constraints inherent to the transport network can make it difficult to realize this potential. As an example, network constraints can give rise to hidden monopolies, where even a small firm can exhibit market dominance because of its position in the network.

The dangers of such hidden monopolies are especially salient (and the corresponding efficiency loss is especially large) in the case of electricity markets, since power flows cannot be controlled in an end-to-end manner due to Kirchhoff's laws. Even though California's electricity crisis is long past, examples of generators attempting to exploit this sort of market power are still common today, e.g., JP Morgan was fined $410 million for market manipulations in California and the midwest from September 2010 to November 2012 [FER13].

Contributions of this paper

Our goal in this paper is to provide insight into the design (and regulation) of market makers that govern transport in networked marketplaces. In particular, we study a model of networked Cournot competition in which transport between geographically distinct markets is governed by a market maker (market operator) and subject to network flow constraints. Our results focus on the impact the design of the market maker has on the equilibrium outcomes of the game between firms and the market maker.

Our first contribution is the model itself. We introduce a general, parameterized model of a market maker (Section 2) in a centrally managed networked Cournot competition. In our model, each market contains multiple firms competing locally in a Cournot competition. The market maker acts as an intermediary between markets by buying from some markets and selling to other markets, using its network to transport the goods between markets, subject to the constraints of the network. The market maker clears the market by maximizing a payoff function that is parameterized by the tradeoff between the benefit to each of the three key parties – the consumer, the producer, and the market maker itself.

Our second contribution is the characterization of the equilibria structure as a function of the design parameters of the market maker (Section 3). Our main result (Theorem 2) highlights a wide variety of behaviors – depending on the design of the market maker, there may be a unique equilibrium, multiple equilibria, or no equilibria. Further, when equilibria do exist, the game may form a weighted potential game or not depending on the design choice. Beyond characterizing existence of equilibria, in the case of linear costs, homogeneous demands, and an unconstrained network, we are able to explicitly characterize the unique equilibrium outcome as a function of the market maker design. This allows us to perform a more detailed study of the impact of the market maker. For example, the characterization highlights that the total production by all firms is independent of the design of the market maker (in this setting), but that the relative production of the firms may vary dramatically depending on the design of the market maker. Additionally, the characterization allows us to provide results highlighting the value of the trade provided by the market maker as well as the efficiency of the market maker (i.e., how close the outcomes of the game are to the outcomes of a single, aggregate Cournot market) as a function of the market maker design.

Our third contribution focuses on the design of the market maker. In particular, we show how to (approximately) optimally design the market maker payoff so as to maximize a desired social/regulatory objective, e.g., social welfare, (Section 4). Our primary tool is the characterization of the equilibria provided in Section 3. Then, we utilize the sum of squares (SOS) relaxation framework to judge the quality of our approximately optimal design choice. The results highlight the, perhaps counterintuitive, observation that if the market maker intends to optimize social welfare, it should not use social welfare as the objective in clearing the market. We further illustrate our proposed approach to market maker design on a stylized example that represents a caricature of the California electricity market. Our results underscore the importance of careful design.

Related literature

Models of competition in networked settings have received considerable attention in recent years. These models come in various forms, including networked Bertrand competition, e.g., [Guz11, AS15, CR08, ABO09], networked Cournot competition, e.g., [BEI14, ABH$^+$14, Ilk09], and various other non-cooperative bargaining games where agents can trade via bilateral contracts and a network determines the set of feasible trades, e.g., [Ell15, CG12, Nav15, AM12, Man11].

Our paper fits into the emerging literature on networked Cournot competition; however our focus and model differ considerably from existing work. In particular, beginning with [BGK85] and continuing through [Ilk09, BEI14, ABH$^+$14], the literature on networked Cournot competition has focused on models where the network structure emerges as a result of firms having a fixed, limited set of markets in which they can participate and participation in these markets is unconstrained and independent of the actions of other firms. In contrast, in our model the network constrains flows between markets, and so there are coupled participation constraints for the firms. Further, the literature on networked Cournot competition has focused on situations where firms operate independently, without governance, while we focus on situations where transport across markets is managed by a market maker.

The line of work that is most relevant to the questions studied in this paper comes from the electricity market literature, where versions of Cournot competition subject to network constraints have been studied for nearly two decades, see [VBRR05] for a survey. In this setting Cournot models often provide good explanations for observed price variations [WRW09], and so are quite popular. For example, Cournot models have been applied to perform detailed studies of electricity markets in the US [BB99], Scandinavia [AB95], Spain [AONMB99, RVR99], and New Zealand [Sco98, SR96], among others.

Due to the importance of the ISO in electricity markets, papers within this literature often include a model of a market maker, e.g., [Wil02, YOA04, MHP03, JYS99, Hob01, YAO08, BCLW14]. However, with rare exception, these papers focus on a market maker that is regulated to maximize social welfare, and thus do not explore the impact of differing market maker payoffs, nor how to design the market maker to optimize a particular social objective. Further, these papers focus exclusively on detailed models of power flows, and thus do not apply to more general network models, such as classical flow models, which are relevant to other applications. Our results, on the other hand, apply to networks with general linear constraints, including both linearized power flow constraints and classical network flow constraints.

To the best of our knowledge, this is the first paper to focus on understanding the impact of, and how to optimally design, a market maker that governs transport in a networked marketplace.

2 Model

Our focus is a marketplace where a constrained transport network, operated by a market maker, connects firms and markets. Specifically, we consider an economy dealing in a single commodity that is comprised of a set of markets $\mathcal{M}$, a set of firms $\mathcal{F}$, and a market maker who facilitates transport of the commodity between the markets. Within this setting, we study Cournot competition over the networked markets, considering a static game of complete information among the firms and the market maker.

Each firm $f \in \mathcal{F}$ supplies to exactly one market, denoted by $\mathcal{M}(f)$. Let $\mathcal{F}(m)$ denote the set of firms that supply to market $m \in \mathcal{M}$. Denote the supply of firm $f \in \mathcal{F}$ to market $\mathcal{M}(f)$ by $q_f \in \mathbb{R}_+$, and let $\boldsymbol{q} := (q_f, f \in \mathcal{F}) \in \mathbb{R}_+^{|\mathcal{F}|}$ denote the vector of supplies of all firms in $\mathcal{F}$. Additionally, for each $f \in \mathcal{F}$, let $\boldsymbol{q}_{-f}$ denote the vector of supplies of all firms in $\mathcal{F}$, except f. The cost incurred by firm $f \in \mathcal{F}$ for producing $q_f \in \mathbb{R}_+$ is $c_f(q_f)$. Assume $c_f : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, convex, twice continuously differentiable, and $c_f(0) = 0$.

Crucially, the production of each firm in our model can be reallocated to other markets by a market maker that controls a constrained transport network. We consider a single market maker that facilitates transport of the commodity between markets. The market maker can procure supply from one market and transport it to a different market, subject to network constraints. Denote the quantity supplied by the market maker to market $m \in \mathcal{M}$ by r_m. Our convention is that $r_m \geq 0$ ($r_m < 0$) denotes a net supply (net demand) of the commodity by the market maker in market m. For convenience, let $\boldsymbol{r} := (r_m,\ m \in \mathcal{M}) \in \mathbb{R}^{|\mathcal{M}|}$ denote the vector of supplies by the market maker. Since the market maker only transports the commodity, the market maker neither consumes nor produces. So, we have $\mathbf{1}^\top \boldsymbol{r} = 0$, where $\mathbf{1}$ is a vector of ones with dimension $|\mathcal{M}|$.[1]

The reallocation of supply by the market maker, $\boldsymbol{r}$, is subject to the flow constraints of the network. We model these constraints by restricting $\boldsymbol{r}$ to a polyhedral set $\mathcal{P} := \left\{\boldsymbol{r} \in \mathbb{R}^{|\mathcal{M}|} \ : \ \boldsymbol{A}\boldsymbol{r} \leq \boldsymbol{b}\right\} \subseteq \mathbb{R}^{|\mathcal{M}|}$, where $\boldsymbol{A}$ and $\boldsymbol{b}$ define the half-spaces of $\mathcal{P}$. This formulation can capture constraints in traditional flow networks, as well as power flow constraints arising from linearized Kirchoff's laws and line limits. We remark that our results can be generalized to $\mathcal{P}$ being a general convex semi-

[1] We recognize that, in some cases, the market maker may have an incentive to dispose off some of its purchases. We can model such behavior by replacing the constraint $\mathbf{1}^\top \boldsymbol{r} = 0$ with $\mathbf{1}^\top \boldsymbol{r} \leq 0$. Most of our results continue to hold with the latter constraint. However, our motivating application of electricity markets does not feature disposal; hence, we assume $\mathbf{1}^\top \boldsymbol{r} = 0$.

algebraic set with nonempty interior.

The price at each market in the network is dependent on both the production of the firms and the reallocation performed by the market maker. As is traditional when studying Cournot competition, we focus on the case of linear inverse demand functions. In particular, assume that the price p_m, in each market $m \in \mathcal{M}$, has the form

$$p_m(d_m) := \alpha_m - \beta_m d_m, \tag{1}$$

for some $\alpha_m, \beta_m > 0$. Here, d_m is the aggregate demand in market m. Importantly, the aggregate demand in each market is determined by both the actions of the firms and the market maker, i.e., $d_m = r_m + \sum_{f \in \mathcal{F}(m)} q_f$.

Given the prices in each market, p_m, we can write the payoff functions for the firms and the market maker. The payoff of firm $f \in \mathcal{F}$ is given by its profit, defined as

$$\pi_f(\boldsymbol{q}, \boldsymbol{r}) := q_f \cdot p_{\mathcal{M}(f)} \left(r_{\mathcal{M}(f)} + \sum_{f' \in \mathcal{F}(\mathcal{M}(f))} q_{f'} \right) - c_f(q_f). \tag{2}$$

Thus, firm f maximizes $\pi_f(\boldsymbol{q}, \boldsymbol{r})$ over $q_f \in \mathbb{R}_+$, given $(\boldsymbol{q}_{-f}, \boldsymbol{r})$.

For the market maker, the payoff function is a design choice. In many regulated settings, e.g., electricity markets, it is common for the market maker to optimize some metric of social benefit. Our goal in this paper is to explore the impact of the market maker payoff functions, and so we focus on a broad, parameterized class of maker maker payoff functions defined as follows. Given $\boldsymbol{q}$, the market maker maximizes $\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ over $\boldsymbol{r} \in \mathcal{P}$ and $\mathbf{1}^\top \boldsymbol{r} = 0$, where

$$\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) := \sum_{m \in \mathcal{M}} \left[\theta_C \cdot \mathsf{CS}_m(\boldsymbol{q}, \boldsymbol{r}) + \theta_P \cdot \mathsf{PS}_m(\boldsymbol{q}, \boldsymbol{r}) + \theta_M \cdot \mathsf{MS}_m(\boldsymbol{q}, \boldsymbol{r}) \right]. \tag{3}$$

In $\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$, the design parameter $\boldsymbol{\theta} := (\theta_C, \theta_P, \theta_M)^\top \in \mathbb{R}^3_+$ allows the designer to weigh the importance of the following terms, for each $m \in \mathcal{M}$:[2]

$$\mathsf{CS}_m(\boldsymbol{q}, \boldsymbol{r}) := \int_0^{r_m + \sum_{f \in \mathcal{F}(m)} q_f} p_m(w_m)\, dw_m - \left(r_m + \sum_{f \in \mathcal{F}(m)} q_f \right) \cdot p_m \left(r_m + \sum_{f \in \mathcal{F}(m)} q_f \right);$$

$$\mathsf{PS}_m(\boldsymbol{q}, \boldsymbol{r}) := \left(\sum_{f \in \mathcal{F}(m)} q_f \right) \cdot p_m \left(r_m + \sum_{f \in \mathcal{F}(m)} q_f \right) - \sum_{f \in \mathcal{F}(m)} c_f(q_f);$$

$$\mathsf{MS}_m(\boldsymbol{q}, \boldsymbol{r}) := r_m \cdot p_m \left(r_m + \sum_{f \in \mathcal{F}(m)} q_f \right),$$

The quantities CS_m, PS_m, and MS_m admit natural interpretations. Namely, CS_m equals the consumer surplus in market m, PS_m equals the collective producer surplus of all firms supplying in market m, and MS_m equals the merchandizing surplus of the market maker by supplying in market m.

The parameterized class of market maker payoff functions defined in (3) encompasses a wide class of common objectives. To illustrate a few, consider the following definitions.

$$\boldsymbol{\theta}^{\mathsf{SW}} := (1,1,1), \quad \boldsymbol{\theta}^{\mathsf{CS}} := (1,0,0), \quad \boldsymbol{\theta}^{\mathsf{RSW}} := (1,0,1), \quad \boldsymbol{\theta}^{\mathsf{MS}} := (0,0,1). \tag{4}$$

The market maker's payoff function with $\boldsymbol{\theta}^{\mathsf{SW}}$ as the design parameter is the Walrasian social welfare that is widely used in many centrally managed networked marketplaces, including wholesale electricity markets. In the same vein, $\Pi\left(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{\theta}^{\mathsf{CS}}\right)$ is the collective consumer surplus across all markets, and hence, defines a pro-consumer design choice by the market maker. Another common choice is $\Pi\left(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{\theta}^{\mathsf{RSW}}\right)$, the *residual social welfare,* which equals the Walrasian social welfare less the collective producer surplus of all firms. By maximizing the residual social welfare, one hopes that the market maker strikes a balance in optimizing the components of the Walrasian social welfare that do not accrue to firms. In contrast with $\boldsymbol{\theta}^{\mathsf{SW}}$, $\boldsymbol{\theta}^{\mathsf{CS}}$, and $\boldsymbol{\theta}^{\mathsf{RSW}}$, the choice of $\boldsymbol{\theta}^{\mathsf{MS}}$ as the design parameter defines a profit-maximizing market maker.

Note that the class of payoffs we consider does not account for any variable costs associated with transporting the commodity through the network. However, as long as the variable costs are convex in $\boldsymbol{r}$, most of our results continue to hold.

A motivating example: Many networked marketplaces with market makers that govern transport can be described by the model discussed above, but to provide a concrete motivating example for use throughout this paper, we consider the case of *wholesale electricity markets*. We illustrate our results with this example in Section 4.3.

Most electricity markets in the US are managed by a regulatory entity known as an Independent

[2]The notation $\boldsymbol{\theta} \in \mathbb{R}_+^3$ should be understood to mean that $\boldsymbol{\theta} \gneq \mathbf{0}$ since the market maker's payoff becomes zero trivially when $\boldsymbol{\theta} = \mathbf{0}$.

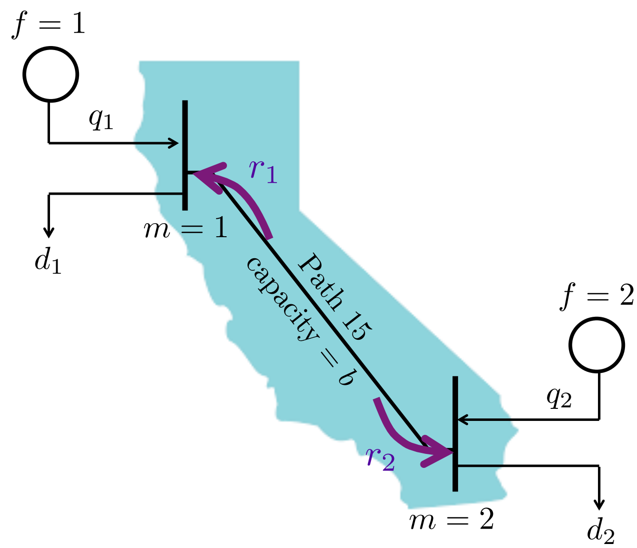


Figure 1: Example of a two-market two-firm networked marketplace. This example reperesents a caricature of the wholesale electricity market in California. Here, northern and southern California are represented as two nodes connected by a transmission line - Path 15 - that is often congested (see [Swe08]).

System Operator (ISO). The role of the ISO is to facilitate efficient exchange of power between supply and demand while ensuring that power flows through the network satisfy the operating constraints of the grid. Thus, the ISO plays the role of the market maker in our model.

To illustrate the model, consider the two-node network in Figure 1. Here, northern and southern California are modeled as two nodes connected by a transmission line – Path 15. Assume, for simplicity, that there is one generator at each node and the transmission line has a capacity $b \in \mathbb{R}_+$. The California Independent System Operator (CAISO) serves as the market maker, governing transport, and seeks to maximize social welfare through reallocating generation.

We can model the strategic interactions in this simple example as a game where, there are two markets $\mathcal{M} = \{1, 2\}$ with inverse linear demand functions $p_1(d_1) = \alpha_1 - \beta_1 d_1$ and $p_2(d_2) = \alpha_2 - \beta_2 d_2$, and two firms $\mathcal{F}(1) = \{1\}$ and $\mathcal{F}(2) = \{2\}$ with cost functions $c_1(q_1)$ and $c_2(q_2)$, respectively. The set of feasible reallocations by the market maker is $\mathcal{P} = \{\boldsymbol{r} \in \mathbb{R}^2 : |r_1| \le b,\ |r_2| \le b\}$. The market maker's payoff is the social welfare, i.e., the design parameter is $\boldsymbol{\theta}^{\mathsf{SW}}$.

Equilibrium definition: We conclude this section by formally describing the networked Cournot competition as a game, denoted by $\mathcal{G}(\boldsymbol{\theta})$, where the *players* include the collection of firms $\mathcal{F}$ and the market maker, the *strategy sets* are defined by $q_f \in \mathbb{R}_+$ for each $f \in \mathcal{F}$ and $\boldsymbol{r} \in \mathcal{P}$ and $\mathbf{1}^\top \boldsymbol{r} = 0$, and the *payoff functions* are π_f for each firm $f \in \mathcal{F}$ and Π for the market maker.

We focus our analysis on the Nash equilibria outcomes, which are defined as follows: $(\boldsymbol{q}, \boldsymbol{r}) \in$

$\mathbb{R}_+^{|\mathcal{F}|} \times \mathcal{P}$, satisfying $\mathbf{1}^\top \boldsymbol{r} = 0$, comprises a *Nash equilibrium* of $\mathcal{G}(\boldsymbol{\theta})$, if

$$\pi_f(\boldsymbol{q}, \boldsymbol{r}) \geq \pi_f(q_f', \boldsymbol{q}_{-f}, \boldsymbol{r}) \text{ for all } q_f' \in \mathbb{R}_+,$$
$$\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) \geq \Pi(\boldsymbol{q}, \boldsymbol{r'}; \boldsymbol{\theta}), \text{ for all } \boldsymbol{r'} \in \mathcal{P},\ \mathbf{1}^\top \boldsymbol{r'} = 0.$$

The effect of $\boldsymbol{\theta}$ on the Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ represents the impact of market maker design, and is the focus of the current paper.

3 Characterizing the Nash Equilibria

In this section, we describe our first set of results, which provide characterizations of the equilibria outcomes, and contrast the equilibrium in our networked Cournot marketplace to non-networked Cournot models. Then, in Section 4, we use the characterizations provided here to inform the design of the market maker.

3.1 Existence and uniqueness

Classical Cournot competition among a set of firms in a single market with inverse linear demand functions is known to be a potential game (see [Sla94] and [MS96]) and, recently, this property has been shown to extend to a form of networked Cournot competition, as shown in [ABH+14]). These characterizations are powerful, since they allow results about equilibrium existence and uniqueness to be derived through analysis of the underlying potential function of the game. However, the results in [ABH+14] focus on a form of networked competition over bipartite graphs with no market maker; thus they do not apply to the model we consider here. But, given the results for these classical and networked Cournot models, an optimistic reader expect a similar conclusion for the model we consider. In the results that follow, we show that this is true in some situations – under some assumptions, we show that the model we consider yields a weighted potential game – however, the structure of the game is more complex in general.

Before stating our results, we begin by formally defining the notion of a weighted potential game. Consider a game $\mathcal{G}$ with players $1, \ldots, N$, actions $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$ for $i = 1, \ldots, N$, where $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathcal{X} \subseteq \mathbb{R}^{n_1 + \ldots + n_N}$, and payoff functions $\varphi_i : \mathbb{R}^{n_1 + \ldots + n_N} \to \mathbb{R}$ for each player $i = 1, \ldots N$. The game $\mathcal{G}$ is said to be a *weighted potential game*, if there exists a vector of weights $w \in \mathbb{R}_{++}^N$ and a potential function $\Phi : \mathbb{R}^{n_1 + \ldots + n_N} \to \mathbb{R}$ such that, for each $i = 1, \ldots, N$,

$$\Phi(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) - \Phi(\boldsymbol{x}_i', \boldsymbol{x}_{-i}) = w_i \cdot \left[\varphi_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) - \varphi_i(\boldsymbol{x}_i', \boldsymbol{x}_{-i})\right] \tag{5}$$

for each $(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) \in \mathcal{X}$ and $(\boldsymbol{x}_i', \boldsymbol{x}_{-i}) \in \mathcal{X}$.

Our first result highlights that, for some design parameters $\boldsymbol{\theta}$, the model of networked competition we consider is a weighted potential game with a potential function that can be represented as a perturbed version of the market maker payoff.

Theorem 1. *If $\theta_M + \theta_P - \theta_C > 0$, then $\mathcal{G}(\boldsymbol{\theta})$ is a weighted potential game with the potential function $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$, given by*

$$\begin{aligned}\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) := \Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - (\theta_M - \theta_P) \sum_{m \in \mathcal{M}} \frac{\beta_m}{2} \left(\sum_{f \in \mathcal{F}(m)} q_f \right)^2 \\ - \sum_{f \in \mathcal{F}} \left[(\theta_M + \theta_P - \theta_C) \frac{\beta_{\mathcal{M}(f)}}{2} q_f^2 + (\theta_C - \theta_M) \left(\alpha_{\mathcal{M}(f)} q_f - c_f(q_f) \right) \right].\end{aligned} \tag{6}$$

A proof of this result is provided in Appendix A.1. The fact that $\mathcal{G}(\boldsymbol{\theta})$ is a weighted potential game highlights that the game has a number of favorable properties. In particular, maximizers of $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ over the joint strategy set of $\mathcal{G}(\boldsymbol{\theta})$ are Nash equilibria, and this fact can be used to infer the existence of Nash equilbria. Further, strict concavity of $\hat{\Pi}$ can be leveraged to conclude the uniqueness of Nash equilibrium, that is characterized as the solution to:

$$\begin{aligned} \mathcal{C}(\boldsymbol{\theta}) \ : \underset{\boldsymbol{q}, \boldsymbol{r}}{\text{maximize}} \quad & \hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}), \\ \text{subject to} \quad & \boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}, \boldsymbol{r} \in \mathcal{P}, \ \mathbf{1}^\top \boldsymbol{r} = 0. \end{aligned} \tag{7}$$

In addition, if cost functions are increasing linear functions or convex quadratic functions, there exists a unique equilibrium that can be found efficiently. Finally, many natural learning dynamics are guaranteed to converge to an equilibrium in potential games. See [MS96] and more recent publications, e.g., [FL98, You04, SA04, MAS09, MYAS09] for a comprehensive discussion on the topic.

However, the characterization of existence provided by Theorem 1 is not complete. It turns out that, for many design parameters, the structure of the game is more complex and, in particular, the game is not a weighted potential game. Despite this, in such cases a Nash equilibrium may still be guaranteed to exist. The theorem below provides a more complete view of equilibrium existence and uniqueness.

Theorem 2. *Suppose $\mathcal{P}$ is compact, and let*

$$\gamma := 1 - \min_{m \in \mathcal{M}} \left(1 + \sum_{f \in \mathcal{F}(m)} \frac{\beta_m}{\beta_m + \inf_{q_f \geq 0} c''_f(q_f)} \right)^{-1}. \tag{8}$$

(a) If $2\theta_M - \theta_C \geq 0$ or $\theta_M + \theta_P - \theta_C > 0$, then $\mathcal{G}(\boldsymbol{\theta})$ has a Nash equilibrium.

(b) If $2\theta_M - \theta_C \geq \gamma \cdot (\theta_M + \theta_P - \theta_C) > 0$, then the set of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ is nonempty, and is

identical to the set of optimizers of $\mathcal{C}(\boldsymbol{\theta})$. Furthermore, if the inequalities are strict, then $\mathcal{G}(\boldsymbol{\theta})$ has a unique Nash equilibrium.

The formal proof is deferred till Appendices A.2 and A.3. The argument hinges on a result due to [Ros65]. It relies on the market maker's payoff $\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ being continuous in $\boldsymbol{q}$ and concave in $\boldsymbol{r}$. In essence, $\mathcal{G}(\boldsymbol{\theta})$ has additional structure for design parameters even beyond where it is a potential game. Some insight for the form of the conditions in the theorem can be understood from the proof. In particular, the market maker's payoff function $\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ is concave in $\boldsymbol{r}$ if and only if $2\theta_M - \theta_C \geq 0$. Additionally, from Theorem 1, we know that $\mathcal{G}(\boldsymbol{\theta})$ is a potential game with $\hat{\Pi}$ as the potential function when $\theta_M + \theta_P - \theta_C > 0$. Finally, $2\theta_M - \theta_C \geq \gamma \cdot (\theta_M + \theta_P - \theta_C) > 0$ implies that $\hat{\Pi}$ is concave, and it is strictly concave when the inequality is strict. Finally, the concavity of $\hat{\Pi}$, together with Neyman's result [Ney97], yields the equality between the sets of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ and the optimizers of $\mathcal{C}(\boldsymbol{\theta})$.

In Figure 2, we visualize the regions defined by the conditions in Theorem 2. Note that the Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ is invariant under a positive scaling of $\boldsymbol{\theta}$; thus it suffices to consider $\boldsymbol{\theta} \in \mathbb{R}^3_+$ for which $\theta_C + \theta_P + \theta_M = 1$, i.e., $\boldsymbol{\theta}$ lies on the 3-dimensional simplex. Hereafter, denote by Δ the 3-dimensional simplex. Notice that the conditions on $\boldsymbol{\theta}$ in Theorem 2(b) depend on γ, which in turn is a function of the inverse linear demand functions in the markets and the cost functions of the firms. Since costs are convex, c_f'' is nonnegative, and thus,

$$\gamma \leq \max_{m \in \mathcal{M}} \frac{|\mathcal{F}(m)|}{1 + |\mathcal{F}(m)|} < 1.$$

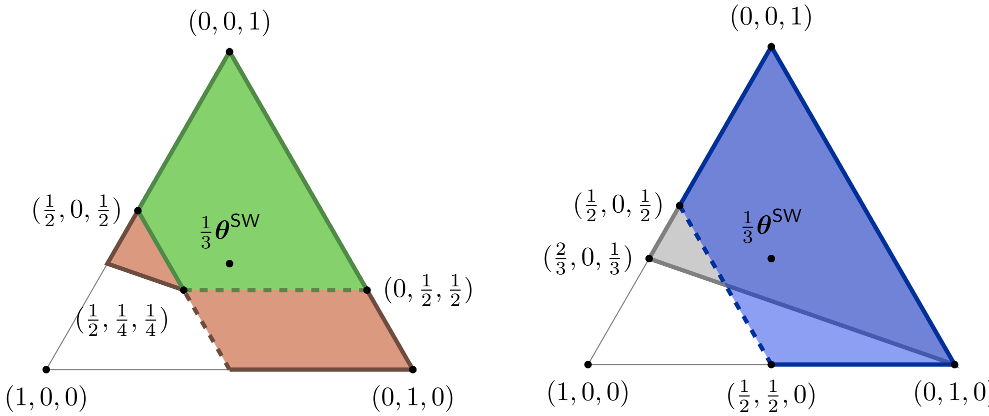
Note that $\gamma = \frac{1}{2}$ when each market has only one firm and costs are increasing linear functions. For illustrative purposes, we choose $\gamma = \frac{1}{2}$ to portray the various regions of Δ in Figure 2, where $\mathcal{G}(\boldsymbol{\theta})$ has different properties.

Theorems 1 and 2 provide sufficient conditions for equilibrium existence and uniqueness, but do not address the question of necessity or tightness. To provide some insight into necessity, we provide examples to highlight that each of the properties may fail to hold if the respective conditions are not met. The examples are all constructed using the simple two-market two-firm example in Section 2. Further, they all focus on only the case when each firm has increasing linear costs and both markets have identical inverse linear demand functions.

The Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ in this restricted setting can be explicitly computed for all $\boldsymbol{\theta} \in \mathbb{R}^3_+$. The results are derived in Lemmas 2, 3, and 4 in Appendix B. Using these results, we construct examples of $\boldsymbol{\theta}$ in Appendix B to illustrate the following:

1. When neither $\theta_M + \theta_P - \theta_C > 0$ nor $2\theta_M \geq \theta_C$ holds, a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ may not exist.

2. When $2\theta_M \geq \theta_C$, but not $\theta_M + \theta_P - \theta_C > 0$, a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ exists, but $\mathcal{G}(\boldsymbol{\theta})$ is not



(a) Existence and uniqueness (b) Concave v/s potential games

Figure 2: (a) An illustration of Theorem 2 for $\boldsymbol{\theta} \in \Delta$. A Nash equilibrium may not exist in the unshaded region, it exists but may not be unique in the brown region, and it is unique and is given by the unique optimizer of $\mathcal{C}(\boldsymbol{\theta})$ in the green region. (b) An illustration of Theorem 2(a) for $\boldsymbol{\theta} \in \Delta$. The grey region is defined by $2\theta_M - \theta_C \geq 0$, where a Nash equilibrium exists owing to a variant of $\mathcal{G}(\boldsymbol{\theta})$ being a concave game. The blue region is defined by $\theta_M + \theta_P - \theta_C > 0$, where a Nash equilibrium exists because $\mathcal{G}(\boldsymbol{\theta})$ is a potential game. Dotted line segments on the boundaries of various sets do not belong to the respective sets.

a weighted potential game.

3. When $2\theta_M - \theta_C \geq 0$, and $\theta_M + \theta_P - \theta_C > 0$, but $2\theta_M - \theta_C \geq \gamma \cdot (\theta_M + \theta_P - \theta_C)$ does not hold, a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ may exist that is not an optimizer of $\mathcal{C}(\boldsymbol{\theta})$.

4. When $2\theta_M - \theta_C = \gamma\,(\theta_M + \theta_P - \theta_C) > 0$, then $\mathcal{G}(\boldsymbol{\theta})$ may have a multitude of Nash equilibria, all of which are optimizers of $\mathcal{C}(\boldsymbol{\theta})$.

3.2 Example with linear costs and homogeneous demands

To this point our results have focused only on existence and uniqueness. We now provide a more detailed characterization of the equilibria. Specifically, our goal is to study the parametric dependence of the Nash equilibria on $\boldsymbol{\theta}$.

Without making stronger assumptions on the nature of the game, such a characterization is difficult. To allow interprebility of the results, we focus on a restricted setting where each market has a single firm with linear increasing cost and the markets have identical linear demand functions. Additionally, we focus on the case of an unconstrained network, i.e., $\mathcal{P} = \mathbb{R}^{|\mathcal{M}|}$. It is possible to provide a more general characterization at the expense of interprebility.

In this setting, we are able to offer explicit formulae for the unique Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ under a subset of the design parameters in Proposition 1. Importantly, this characterization allows us to contrast the result of competition in the networked marketplace we consider with two cases of particular interest: (a) competition in a collection of non-networked markets, i.e., a setting without transport between markets, and (b) competition in an aggregated market, i.e., a setting where the markets are merged into a single aggregate marketplace without a market maker. The comparison with (a) provides insight into the efficiency of the network and the comparison with (b) provides insight into the efficiency of the market maker.

Consider $\mathcal{G}(\boldsymbol{\theta})$ with: (1) an unconstrained network, $\mathcal{P} := \mathbb{R}^{|\mathcal{M}|}$, (2) a collection of firms $\mathcal{F}$ having linear costs $c_f(q_f) := C_f q_f$ for each $f \in \mathcal{F}$, where $C_f > 0$, (3) a collection of markets $\mathcal{M}$, where a single firm supplies in each market, i.e., $|\mathcal{F}(m)| = 1$, for $m \in \mathcal{M}$, and (4) spatially homogeneous inverse linear demand functions, given by $p_m(d_m) = \alpha - \beta d_m$, for each $m \in \mathcal{M}$, for some $\alpha, \beta > 0$. Define $\boldsymbol{C} := (C_f, f \in \mathcal{F})$. Denote the mean and the standard deviation of the firms' marginal costs by

$$\bar{C} := \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} C_f, \quad \text{and} \quad \sigma_C := \sqrt{\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} (C_f - \bar{C})^2}, \tag{9}$$

respectively. Denote this family of games by $\mathcal{G}^u(\boldsymbol{\theta}; \boldsymbol{C}, \alpha, \beta)$, parameterized by the design parameter, firms' marginal costs, and the parameters defining the identical market demand functions. Then, we have the following result on $\mathcal{G}^u(\boldsymbol{\theta}; \boldsymbol{C}, \alpha, \beta)$.

Proposition 1. *Consider $\mathcal{G}^u(\boldsymbol{\theta}; \boldsymbol{C}, \alpha, \beta)$, where $\bar{C}$ and σ_C are as defined in (9). If $2\theta_M - \theta_C > \frac{1}{2}(\theta_M + \theta_P - \theta_C) > 0$ and $\alpha \geq (1 + \kappa(\boldsymbol{\theta})) \max_{f \in \mathcal{F}} C_f - \kappa(\boldsymbol{\theta}) \bar{C}$, then $\mathcal{G}^u(\boldsymbol{\theta}; \boldsymbol{C}, \alpha, \beta)$ has a unique Nash equilibrium, given by*

$$q_f = \frac{1}{2\beta} \left[\alpha - \bar{C} - (1 + \kappa(\boldsymbol{\theta})) \left(C_f - \bar{C} \right) \right], \tag{10}$$

$$r_{\mathcal{M}(f)} = \frac{\kappa(\boldsymbol{\theta})}{\beta} \left(C_f - \bar{C} \right), \tag{11}$$

for each $f \in \mathcal{F}$, where $\kappa(\boldsymbol{\theta}) := \frac{\theta_M + \theta_P - \theta_C}{3\theta_M - \theta_P - \theta_C}$. Moreover, the Walrasian social welfare at the unique Nash equilibrium is given by

$$\sum_{m \in \mathcal{M}} [\mathsf{CS}_m(\boldsymbol{q}, \boldsymbol{r}) + \mathsf{PS}_m(\boldsymbol{q}, \boldsymbol{r}) + \mathsf{MS}_m(\boldsymbol{q}, \boldsymbol{r})] = \frac{3|\mathcal{F}|}{8\beta} \left[(\alpha - \bar{C})^2 + \sigma_C^2 + \frac{1}{3} \kappa(\boldsymbol{\theta})(6 - \kappa(\boldsymbol{\theta})) \sigma_C^2 \right].$$

A proof is given in Appendix A.4. Note that, because we consider an unconstrained network, our proof technique from Theorem 2(b) no longer applies. Thus, we take a different approach to analyzing $\mathcal{C}(\boldsymbol{\theta})$ focused on the Karush-Kuhn-Tucker (KKT) optimality conditions.

To obtain some insight from Proposition 1, note that equations (10) – (11) reveal that, the production of a firm q_f and the market-maker's supply in the market served by that firm $r_{\mathcal{M}(f)}$, both depend on the marginal cost of the firm C_f relative to the average marginal cost of all firms $\bar{C}$. Under the conditions of Proposition 1, one can show that $\kappa(\boldsymbol{\theta}) > 0$. Hence, the firms' productions are in fact ordered inversely by their marginal costs. Moreover, the market maker buys from markets having firms with lower marginal costs and supplies to markets having firms with higher marginal costs. The total production by all firms, however, is independent of $\boldsymbol{\theta}$, and is given by

$$\sum_{f \in \mathcal{F}} q_f = \frac{|\mathcal{F}|}{2\beta} \left(\alpha - \bar{C}\right). \tag{12}$$

The market maker's design choice only influences the relative production between the firms and the quantities supplied by the market maker to various markets.

Proposition 1 also lets us investigate the efficiency of the equilibria. By studying the effect of the design parameter on the social welfare at the unique Nash equilibrium, we unravel the impact of the design choice on the competitiveness of the market. As we remarked earlier, a popular choice of $\boldsymbol{\theta}$ for a regulated marketplace like the wholesale electricity markets is $\boldsymbol{\theta}^{\mathsf{SW}}$ defined in (4), i.e., the market maker optimizes the social welfare function. Then, $\kappa\left(\boldsymbol{\theta}^{\mathsf{SW}}\right) = 1$. We notice that the social welfare at the unique Nash equilibrium increases with $\kappa(\boldsymbol{\theta})$ over the interval $[1, 3]$. Moreover, it is easy to construct a $\boldsymbol{\theta}$ that satisfies the conditions in Proposition 1 with $1 < \kappa(\boldsymbol{\theta}) < 3$. So, if maximizing the social welfare at the unique Nash equilibrium is indeed the design goal, $\boldsymbol{\theta}^{\mathsf{SW}}$ is *not* the optimal design choice.

This motivates the question of how much efficiency is lost by naively choosing the design parameter $\boldsymbol{\theta}^{\mathsf{SW}}$. To address this, note that the social welfare at the Nash equilibrium with $\boldsymbol{\theta}^{\mathsf{SW}}$ is given by $(\alpha - \bar{C})^2 + \frac{8}{3}\sigma_C^2$. Also, observe that $\kappa(\boldsymbol{\theta})(6 - \kappa(\boldsymbol{\theta})) \leq 9$. Combining these allows us to derive a simple upper bound on the ratio of the largest attainable social welfare at a Nash equilibrium to that obtained with $\boldsymbol{\theta}^{\mathsf{SW}}$.

$$\frac{(\alpha - \bar{C})^2 + \sigma_C^2 + \frac{1}{3}\kappa(\boldsymbol{\theta})(6 - \kappa(\boldsymbol{\theta}))\sigma_C^2}{(\alpha - \bar{C})^2 + \frac{8}{3}\sigma_C^2} \leq \frac{1 + 4\left(\frac{\sigma_C}{\alpha - \bar{C}}\right)^2}{1 + \frac{8}{3}\left(\frac{\sigma_C}{\alpha - \bar{C}}\right)^2} \leq \frac{3}{2}.$$

The last step follows from the fact that $h(x) := (1 + 4x)/\left(1 + \frac{8}{3}x\right)$ is increasing in $x \geq 0$, and is bounded from above by $\lim_{x \to \infty} h(x) = \frac{3}{2}$.

When $\boldsymbol{\theta}$ is varied such that $\kappa(\boldsymbol{\theta})$ increases from one, we have already argued that the social welfare at the Nash equilibrium increases. Who stands to benefit from such an increase? Is it the consumers, the producers, or the market maker? Recall that a metric of consumer benefit is the aggregate consumer surplus $\sum_{m \in \mathcal{M}} \mathsf{CS}_\mathsf{m}(\boldsymbol{q}, \boldsymbol{r})$ at the Nash equilibrium. Similarly, the aggregate producer surplus $\sum_{m \in \mathcal{M}} \mathsf{PS}_\mathsf{m}(\boldsymbol{q}, \boldsymbol{r})$ and the merchandising surplus $\sum_{m \in \mathcal{M}} \mathsf{MS}_\mathsf{m}(\boldsymbol{q}, \boldsymbol{r})$ at the Nash equilibrium measures the benefits to the producers and the market maker, respectively. One can show that the aggregate consumer and producer surpluses both increase, when $\boldsymbol{\theta}$ is changed to increase $\kappa(\boldsymbol{\theta})$ from one. However, the merchandising surplus decreases. Thus, in the framework considered, a design choice that improves the efficiency of the market, does so to the benefit of the consumers and the producers, but at the expense of the market maker.

Comparison with non-networked Cournot

To study the role of the network, we next analyze the same setting, but remove the network. That is, $\mathcal{P}$ only contains the origin. Each firm then effectively competes as a monopoly in its own market, and the market maker plays no role. Define $\mathcal{G}^n(\boldsymbol{C}, \alpha, \beta)$ as the non-networked Cournot competition among a collection of firms $\mathcal{F}$. Here, $\boldsymbol{C}$ again denotes the vector of firms' marginal costs, and identical market demand functions are identified by parameters α, β. We characterize the Nash equilibria of $\mathcal{G}^n(\boldsymbol{C}, \alpha, \beta)$ in the following result.

Proposition 2. *Consider $\mathcal{G}^n(\boldsymbol{C}, \alpha, \beta)$, where $\bar{C}$ and σ_C are as defined in (9). If $\alpha \geq \max_{f\in\mathcal{F}} C_f$, then $\mathcal{G}^n(\boldsymbol{C}, \alpha, \beta)$ has a unique Nash equilibrium, given by*

$$q_f^n = \frac{1}{2\beta}\left(\alpha - C_f\right).$$

Moreover, the Walrasian social welfare at the unique Nash equilibrium is given by

$$\sum_{f\in\mathcal{F}} \left[\int_0^{q_f^n} p_{\mathcal{M}(f)}(w_f)\, dw_f - C_f q_f^n\right] = \frac{3|\mathcal{F}|}{8\beta}\left[\left(\alpha - \bar{C}\right)^2 + \sigma_C^2\right].$$

The proof is straightforward and is omitted. To compare Propositions 1 and 2, assume that α satisfies the conditions required in both results.

Like in the networked marketplace, the production quantities of the firms are ordered inversely by their marginal costs. Also, the total production of the firms at the Nash equilibrium is given by

$$\sum_{f\in\mathcal{F}} q_f^n = \frac{|\mathcal{F}|}{2\beta}\left(\alpha - \bar{C}\right),$$

which, due to (12), happens to be identical to that in the networked marketplace. Thus, the network does not impact the total production of the firms. Instead, the value of the network is reflected in the social welfare at the Nash equilibrium.

It is straightforward to conclude from Propositions 1 and 2 that the social welfare at the Nash equilibrium is *higher* for the networked marketplace for any design choice $\boldsymbol{\theta}$. This aligns with the intuition that a network available for trade improves the efficiency of the marketplace. Recall that in the networked setting, the social welfare at the unique Nash equilibrium is given by $(\alpha - \bar{C})^2 + \sigma_C^2 + \frac{1}{3}\kappa(\boldsymbol{\theta})(6 - \kappa(\boldsymbol{\theta}))\sigma_C^2$. Again, leveraging the fact that $\kappa(\boldsymbol{\theta})(6 - \kappa(\boldsymbol{\theta})) \leq 9$, we obtain the following bound on the ratio of the social welfares in the networked and the non-networked case.

$$\frac{(\alpha-\bar{C})^2+\sigma_C^2+\frac{1}{3}\kappa(\boldsymbol{\theta})(6-\kappa(\boldsymbol{\theta}))\sigma_C^2}{(\alpha-\bar{C})^2+\sigma_C^2} \leq \frac{1+4\left(\frac{\sigma_C}{\alpha-\bar{C}}\right)^2}{1+\left(\frac{\sigma_C}{\alpha-\bar{C}}\right)^2} \leq 4.$$

The last step follows from the fact that $h(x) := (1+4x)/(1+x)$ is increasing in $x \geq 0$, and is bounded from above by $\lim_{x\to\infty} h(x) = 4$.

Comparison with aggregated Cournot

To study the efficiency of the market maker, we next analyze the same setting, but where the firms are aggregated into a single Cournot market. This comparison is motivated by the fact that one may hope an efficient market maker can facilitate trade in order to allow the networked marketplace to behave like a single market – especially when the network is unconstrained.

Recall that in our example, we considered $|\mathcal{M}|$ markets with identical inverse linear demand functions $p_m(d_m) = \alpha - \beta d_m$ for each $m \in \mathcal{M}$. Then, an aggregation of these markets with a collective demand d admits an inverse linear demand function $p(d) = \alpha - \frac{\beta}{|\mathcal{F}|}d$. Denote the aggregated Cournot competition by $\mathcal{G}^a(\boldsymbol{C},\alpha,\beta)$, where $\boldsymbol{C}$ denotes the vector of firms' marginal costs. The following result then characterizes the unique Nash equilibrium of $\mathcal{G}^a(\boldsymbol{C},\alpha,\beta)$.

Proposition 3. *Consider $\mathcal{G}^a(\boldsymbol{C},\alpha,\beta)$, where $\bar{C}$ and σ_C are as defined in (9). If $\alpha \geq (1+|\mathcal{F}|)\max_{f\in\mathcal{F}} C_f - |\mathcal{F}|\bar{C}$, then $\mathcal{G}^a(\boldsymbol{C},\alpha,\beta)$ has a unique Nash equilibrium, given by*

$$q_f^a = \frac{|\mathcal{F}|}{(1+|\mathcal{F}|)\beta}\left[\alpha-\bar{C}-(1+|\mathcal{F}|)\left(C_f-\bar{C}\right)\right].$$

Moreover, the Walrasian social welfare at the unique Nash equilibrium is given by

$$\int_0^{\sum_{f\in\mathcal{F}} q_f^a} p(w)\,dw - \sum_{f\in\mathcal{F}} C_f q_f^a = \frac{|\mathcal{F}|^2(2+|\mathcal{F}|)}{2(1+|\mathcal{F}|)^2\beta}\left[(\alpha-\bar{C})^2+\frac{2(1+|\mathcal{F}|)^2}{2+|\mathcal{F}|}\sigma_C^2\right].$$

A proof can be found in [LS10], and is omitted for brevity. When comparing the results obtained in Proposition 3 to 1 or 2, assume α satisfies the conditions delineated in each result.

As in each case before, the firms' productions in the aggregated Cournot competition are ordered inversely by their marginal costs. However, in this case the total production quantity is different. In particular, we have

$$\sum_{f\in\mathcal{F}} q_f^a = \frac{|\mathcal{F}|^2}{(1+|\mathcal{F}|)\beta}\left(\alpha-\bar{C}\right).$$

Since $\frac{|\mathcal{F}|^2}{1+|\mathcal{F}|} \geq \frac{|\mathcal{F}|}{2}$, it follows from (12) that the total production quantity in the aggregated Cournot competition is no less than that in the networked marketplace with an unconstrained network. Further, the inequality is strict when $|\mathcal{F}| \geq 2$.

Given increased production, it is natural to expect that the social welfare will be larger in the aggregated Cournot market as well. This turns out to be true. Towards comparing the social welfare of the aggregated Cournot case to our networked marketplace with an unconstrained network, notice that (i) $\frac{|\mathcal{F}|^2(2+|\mathcal{F}|)}{2(1+|\mathcal{F}|)^2} \geq \frac{3|\mathcal{F}|}{8}$ for all $|\mathcal{F}| \geq 1$, (ii) $\frac{2(1+|\mathcal{F}|)^2}{2+|\mathcal{F}|} \geq 4$ for all $|\mathcal{F}| \geq 2$, and (iii) $|\mathcal{F}| = 1$ imply $\sigma_C = 0$. These observations, together with $\kappa(\boldsymbol{\theta})\,(6 - \kappa(\boldsymbol{\theta})) \leq 9$, yield

$$\frac{|\mathcal{F}|^2(2+|\mathcal{F}|)}{2\beta(1+|\mathcal{F}|)^2}\left[(\alpha - \bar{C})^2 + \frac{2(1+|\mathcal{F}|)^2}{2+|\mathcal{F}|}\sigma_C^2\right] \geq \frac{3|\mathcal{F}|}{8\beta}\left[(\alpha - \bar{C})^2 + \sigma_C^2 + \frac{\kappa(\boldsymbol{\theta})(6-\kappa(\boldsymbol{\theta}))}{3}\sigma_C^2\right].$$

As a result, the social welfare in the aggregate Cournot model is no less than that in the networked Cournot model for all possible choices of the design parameter. The inequality is strict when $|\mathcal{F}| \geq 2$. Also, $\frac{2(1+|\mathcal{F}|)^2}{2+|\mathcal{F}|} \to \infty$ as $|\mathcal{F}| \to \infty$. Thus, the ratio of equilibrium social welfares in the aggregated market and the unconstrained networked marketplace (with any choice of $\boldsymbol{\theta}$) grows without bound as the number of firms increases. In a sense, the higher the number of firms, the larger the need for transport, leading to a higher efficiency loss due to the market maker's transport.

4 Market Maker Design

The characterization results from the previous section provide the foundation for us to approach the question of market maker design. That is, to engineer the 'right' design parameter $\boldsymbol{\theta}$, when the market maker has a certain design objective. The example considered in Section 3.2 highlights the importance of this task – even in simple settings, using $\boldsymbol{\theta}^{\mathsf{SW}}$ yields suboptimal outcomes if the goal is to optimize social welfare.

Concretely, the contribution of this section is to find an approximation to the optimal design parameter, and leverage a *sum of squares* (SOS) framework to bound the suboptimality of that choice. We illustrate the efficacy of our approach to market maker design on our two-market two-firm example in Figure 1.

4.1 Formulating the market maker design problem

Assume that the design objective of the market maker is given by a polynomial function $g : \mathbb{R}^{|\mathcal{F}|} \times \mathbb{R}^{|\mathcal{M}|} \to \mathbb{R}$. That is, if the market maker owned and operated the firms, it would maximize $g(\boldsymbol{q}, \boldsymbol{r})$ over the joint strategy set, defined by $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}, \boldsymbol{r} \in \mathcal{P}, \mathbf{1}^\top \boldsymbol{r} = 0$. When playing the game with a collection of strategic firms, the market maker would ideally seek a design parameter $\boldsymbol{\theta}$ that maximizes g at the Nash equilibrium outcome of $\mathcal{G}(\boldsymbol{\theta})$. If there are multiple Nash equilibria, one can modify the goal to maximize the *worst case* g over all Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$.

Recall that we can restrict $\boldsymbol{\theta}$ to the 3-dimensional simplex Δ without loss of optimality. However, optimizing $\boldsymbol{\theta}$ over Δ is challenging. A difficulty arises from having to minimize $g(\boldsymbol{q}, \boldsymbol{r})$ over all Nash equilibria $(\boldsymbol{q}, \boldsymbol{r})$ of $\mathcal{G}(\boldsymbol{\theta})$ for any candidate $\boldsymbol{\theta}$. For instance, if $\mathcal{G}(\boldsymbol{\theta})$ has multiple isolated Nash equilibria, such a minimization amounts to solving a combinatorial problem. However, even if $\mathcal{G}(\boldsymbol{\theta})$ has a unique Nash equilibrium, describing said equilibrium is challenging. For example, if the market maker's payoff function is not a concave function of its action, then its optimal strategy cannot be described by first-order conditions alone. Even if it is concave, computing a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ – and hence, computing g for any candidate $\boldsymbol{\theta}$ – is generally challenging.

In light of these challenges, we restrict the search space for $\boldsymbol{\theta}$ to Θ_ε, described by

$$\theta_C, \theta_P, \theta_M \geq 0,\ \theta_C + \theta_P + \theta_M = 1,\ 2\theta_M - \theta_C \geq \varepsilon + \gamma \cdot (\theta_M + \theta_P - \theta_C) \geq (1+\gamma) \cdot \varepsilon$$

for some sufficiently small $\epsilon > 0$. Theorem 2(b) implies that $\mathcal{G}(\boldsymbol{\theta})$ has a unique Nash equilibrium for each $\boldsymbol{\theta} \in \Theta_\varepsilon$ that also equals the unique optimizer of the convex program $\mathcal{C}(\boldsymbol{\theta})$ in (7) with linear constraints. Hence, the unique Nash equilibrium is exactly characterized by the Karush-Kuhn-Tucker conditions for $\mathcal{C}(\boldsymbol{\theta})$. Thus, we can formulate the following market design problem over Θ_ε.

$$\begin{aligned}
\underset{\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\theta}}{\text{maximize}} \quad & g(\boldsymbol{q}, \boldsymbol{r}), \\
\text{subject to} \quad & \boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}, \boldsymbol{A}'\boldsymbol{r} \leq \boldsymbol{b}', \\
& \boldsymbol{\mu} \in \mathbb{R}_+^{|\mathcal{F}|}, \boldsymbol{\lambda} \in \mathbb{R}_+^{\dim(\boldsymbol{b}')}, \boldsymbol{\theta} = (\theta_C, \theta_P, \theta_M) \in \Delta, \\
& \nabla_{\boldsymbol{r}} \left[\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) + \boldsymbol{\lambda}^\top \left(\boldsymbol{b}' - \boldsymbol{A}'\boldsymbol{r}\right)\right] = 0, \\
& \nabla_{\boldsymbol{q}} \left[\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) + \boldsymbol{\mu}^\top \boldsymbol{q}\right] = 0, \\
& \boldsymbol{\mu}^\top \boldsymbol{q} = 0,\ \boldsymbol{\lambda}^\top \left(\boldsymbol{b}' - \boldsymbol{A}'\boldsymbol{r}\right) = 0, \\
& 2\theta_M - \theta_C \geq \varepsilon + \gamma \cdot (\theta_M + \theta_P - \theta_C) \geq (1+\gamma) \cdot \varepsilon,
\end{aligned} \tag{13}$$

where $\boldsymbol{A}' = \left(\boldsymbol{A}^\top, \boldsymbol{1}, -\boldsymbol{1}\right)^\top$, $\boldsymbol{b}' = \left(\boldsymbol{b}^\top, 0, 0\right)^\top$, and $\dim(\boldsymbol{b}')$ denotes the dimension of $\boldsymbol{b}'$. Here, $\nabla_{\boldsymbol{x}} h(\boldsymbol{x}, \boldsymbol{y})$ denotes the gradient of a function $h : \mathbb{R}^{m_x + m_y} \to \mathbb{R}$ with respect to $\boldsymbol{x}$. In an optimization problem, if the search space is not closed, an optimizer may not exist. To avoid such technical difficulties, we choose to optimize over the closed subset Θ_ε of the design space where $\mathcal{G}(\boldsymbol{\theta})$ admits a unique Nash equilibrium. Denote by $\boldsymbol{\theta}^*$, an optimizer of (13), that defines an optimal design choice. As we illustrated in Section 3.2 through an example, even if $g(\boldsymbol{q}, \boldsymbol{r}) = \Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}^0)$ for some $\boldsymbol{\theta}^0 \in \Theta_\varepsilon$, the design choice $\boldsymbol{\theta}^0$ may not be optimal, that is, it may not be an optimizer of (13).

4.2 Approximately solving the market maker design problem

The market maker design problem in (13) is a so-called Mathematical Program with Equilibrium Constraints (MPEC). Such problems are nonconvex and hard to solve efficiently in general (see [LPR96, Pie01, OKZ13]). Many heuristic searches have been applied to MPECs; but they often do not come with any optimality guarantees. Instead of using existing heuristics, we provide a scheme to find an approximate solution by exploiting our characterization results, and further bound the approximation quality.

Assume henceforth that the cost functions $c_f, f \in \mathcal{F}$ are polynomial functions for which $\mathcal{C}(\boldsymbol{\theta})$ can be cast as a convex program that is solvable in polynomial time. For example, when said costs are quadratic, $\mathcal{C}(\boldsymbol{\theta})$ can be solved as a convex quadratic program. For each $\boldsymbol{\theta} \in \Theta_\varepsilon$, one can efficiently compute the unique Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ by solving $\mathcal{C}(\boldsymbol{\theta})$. Denote the unique Nash equilibrium by $(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta}))$. Any metaheuristic (e.g., grid search, simulated annealing, Monte-Carlo sampling) can be used to explore the space Θ_ε for the largest $g(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta}))$ to obtain an approximate solution of (13). For this exposition, we choose a finite and uniform discretization of Θ_ε. If $g(\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta}))$ attains its maximum at $\boldsymbol{\theta}_{\max}$ over this finite set, we have

$$g(\boldsymbol{q}(\boldsymbol{\theta}_{\max}), \boldsymbol{r}(\boldsymbol{\theta}_{\max})) \leq g(\boldsymbol{q}(\boldsymbol{\theta}^*), \boldsymbol{r}(\boldsymbol{\theta}^*)),$$

where $\boldsymbol{\theta}^*$ is an optimizer of (13). The difference between the two expressions in the above inequality is a measure of the optimality gap of $\boldsymbol{\theta}_{\max}$.

We next present a hierarchy of successively tighter upper bounds for $g(\boldsymbol{q}(\boldsymbol{\theta}^*), \boldsymbol{r}(\boldsymbol{\theta}^*))$. The upper bound at any level of the hierarchy can be computed in polynomial time, and yields a bound on the optimality gap of $\boldsymbol{\theta}_{\max}$. In presenting this hierarchy, we need the following technical result that allows us to restrict the feasible set of (13) to a compact basic semi-algebraic set. The proof is deferred till Appendix A.5.[3].

Lemma 1. *Suppose $\mathcal{P}$ is compact. Consider the optimization problem (13), defined in the variables $\boldsymbol{z} := (\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\theta})$. There exists a compact set $\mathcal{Z} := \{\boldsymbol{z} : h_i(\boldsymbol{z}) \geq 0, i = 1, \ldots, I\}$, where $h_i, i = 1, \ldots, I-1$ are polynomial functions, and $h_I(\boldsymbol{z}) = \bar{Z} - \|\boldsymbol{z}\|_2$ for some constant $\bar{Z} \in \mathbb{R}_+$, such that the feasible set of (13) can be restricted to $\mathcal{Z}$ without loss of optimality.*

With a slight abuse of notation, we use $g(\boldsymbol{z})$ to denote $g(\boldsymbol{q}, \boldsymbol{r})$. Equipped with Lemma 1, (13) can be reformulated as a polynomial optimization problem that seeks to minimize $t \in \mathbb{R}$, subject to $g(\boldsymbol{z}) \leq t$ for all $\boldsymbol{z} \in \mathcal{Z}$. Our upper bounds are then given by the so-called *Lasserre hierarchy* to this polynomial optimization problem (see [Las09, Chapter 4], [Sch05, DKL11]).

[3] If $\mathcal{P}$ is a general compact convex semi-algebraic set with a nonempty interior, our proof technique does not yield a uniform bound on the Lagrange multipliers. One can then use Fritz-John optimality conditions in place of KKT optimality conditions for $\mathcal{C}(\boldsymbol{\theta})$ as constraints in (13), for which the corresponding multipliers can be shown to be bounded. See [JLLP15] for details.

A polynomial is said to be *sum of squares*, denoted SOS, if it can be expressed as a sum of other squared polynomials. For a positive integer d such that $2d \geq \max_{i=1,\ldots,I} \deg(h_i)$, consider the following optimization problem and its optimal value.

$$\begin{aligned} v_d^* := \underset{t,\sigma_0,\ldots,\sigma_I}{\text{minimize}} \quad & t, \\ \text{subject to} \quad & t = g + \sigma_0 + \sigma_1 h_1 + \ldots, \sigma_I h_I, \\ & \deg(\sigma_0) \leq 2d,\ \deg(\sigma_i h_i) \leq 2d,\ i = 1, \ldots, I, \\ & t \in \mathbb{R},\ \sigma_0, \ldots, \sigma_I \text{ are SOS.} \end{aligned} \tag{14}$$

The convergence of the Lasserre hierarchy, as given by [Las09, Theorem 4.1], yields $v_d^* \downarrow g(\boldsymbol{q}(\boldsymbol{\theta}^*), \boldsymbol{r}(\boldsymbol{\theta}^*))$. That is, v_d^* approaches $g(\boldsymbol{q}(\boldsymbol{\theta}^*), \boldsymbol{r}(\boldsymbol{\theta}^*))$ monotonically from above. Furthermore, the determination of whether a polynomial with degree $\leq 2d$ is SOS can be written as a linear matrix inequality in the coefficients of that polynomial (see [Sch05]). Therefore, (14) can be reformulated as a semidefinite program, that is solvable in polynomial time. If $\boldsymbol{\theta}_{\max}$ is a candidate approximate solution for (13), then $v_d^* - g(\boldsymbol{q}(\boldsymbol{\theta}_{\max}), \boldsymbol{r}(\boldsymbol{\theta}_{\max}))$ defines a bound on the optimality gap of $\boldsymbol{\theta}_{\max}$.

4.3 Returning to our motivating example

Consider again the two-market two-firm example discussed in Section 2 that is motivated by the California electricity market. Assume that the design objective is social welfare, i.e., $g(\boldsymbol{q}, \boldsymbol{r}) = \Pi\left(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}^0\right)$, where $\boldsymbol{\theta}^0 := \frac{1}{3}\boldsymbol{\theta}^{\mathsf{SW}}$. In Section 3.2, we argued that $\boldsymbol{\theta}^0$ is not the optimal choice for such a design objective. In the following, we apply our approximation scheme towards choosing the design parameter to maximize $g(\boldsymbol{q}, \boldsymbol{r})$ at the equilibrium.

Consider a particular setting where: the two markets have identical linear inverse demand functions $p_m(d_m) := 1 - d_m$ for each $m \in \{1, 2\}$, the firms have linear costs $c_1(q_1) := \frac{1}{2}q_f$ and $c_2(q_2) := \frac{1}{4}q_2$, and the capacity of the line is $b = \frac{1}{2}$. As a benchmark for comparison, consider $\mathcal{G}(\boldsymbol{\theta})$ with $\boldsymbol{\theta} = \boldsymbol{\theta}^0$, which represents the current practice in electricity markets. Our results in Appendix B indicate that $q_1 = \frac{3}{16}$, $q_2 = \frac{7}{16}$, $r_1 = -r_2 = \frac{1}{8}$ defines the unique Nash equilibrium of $\mathcal{G}\left(\boldsymbol{\theta}^0\right)$ with a social welfare of $\frac{83}{256} \approx 0.324$.

Now, let us design $\boldsymbol{\theta}$ using our approximation scheme. Assume $\varepsilon = 0.001$ and discretize the set Θ_ε as follows. Tile the triangle in Fig. 2 by squares (aligning with the base) with sides that are one-tenth the length of the base. Solve $\mathcal{C}(\boldsymbol{\theta})$ as a convex quadratic program at the vertices of the square tiles that satisfy $\boldsymbol{\theta} \in \Theta_\varepsilon$. Upon maximizing $g(\boldsymbol{q}, \boldsymbol{r}) = \Pi\left(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}^0\right)$ at the solutions of $\mathcal{C}(\boldsymbol{\theta})$ over this discrete set, we obtain $\boldsymbol{\theta}_{\max} = (0.027, 0.627, 0.346)^\top$. And, the social welfare at the unique Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta}_{\max})$ is 0.339, which is higher than 0.324 obtained at that of $\mathcal{G}\left(\boldsymbol{\theta}^0\right)$. One can show that, while the total production from the two firms is identical for both design choices, the cheaper firm produces a larger share with $\boldsymbol{\theta}_{\max}$ than with $\boldsymbol{\theta}^0$.

To gauge the suboptimality of our design choice $\boldsymbol{\theta}_{\max}$, we obtain an upper bound $v_1^* = 0.340$ on the maximum attainable social welfare at a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ over Θ_ε.[4] The upper bound is remarkably close to the social welfare obtained with $\boldsymbol{\theta}_{\max}$. Hence, our design choice $\boldsymbol{\theta}_{\max}$ has a provably good approximation quality.

5 Conclusion

This work considers a Cournot competition of a single commodity among a collection of strategic firms in a centrally managed networked marketplace. The central manager (market maker) facilitates transport of the commodity over an underlying network. The case of wholesale electricity markets is used throughout as our motivating example; our analysis, however, applies more generally to shared economies, supply chains, etc. Of particular interest is understanding the role of the market maker design. That is, we study how the market clearing rule of the market maker influences the Nash equilibrium outcomes of the marketplace.

Our main result (Theorem 2) characterizes the equilibria outcomes over a parameterized family of market maker designs. We identify the set of design parameters over which an equilibrium is guaranteed to exist and is unique. Then, we exploit our characterization to propose an approach for finding an approximately optimal design choice when the market maker has a specific design objective in mind. A sum of squares based relaxation framework is utilized to bound the optimality gap of our approach.

We illustrate our results on a two-node network where the market maker maximizes social welfare to clear the market. We demonstrate that maximizing the social welfare as the market clearing rule is not always an optimal design choice when the objective is to maximize social welfare at the outcome of the game. For the example considered, our approximation scheme in fact yields a near optimal design choice.

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A Proofs

In this section, we prove Theorems 1, 2, and Proposition 1. We begin by defining some notation important in the sequel. For a symmetric matrix $\boldsymbol{X}$, let $\boldsymbol{X} \preceq 0$ (resp. $\boldsymbol{X} \prec 0$) denote that $\boldsymbol{X}$ is negative semidefinite (resp. negative definite). For a finite index set $\mathcal{I}$, let $(x_i, i \in \mathcal{I})$ define a vertical vector concatenation of x_i's, for which $i \in \mathcal{I}$. For an arbitrary function $h : \mathbb{R}^m \to \mathbb{R}$, define $\frac{\partial}{\partial x_i} h \Big|_{\boldsymbol{x}=\boldsymbol{x}_0}$ as the partial derivative of h with respect to x_i at $\boldsymbol{x}_0 \in \mathbb{R}^m$ for $i = 1, \ldots, m$. Further, let $\frac{\partial^2}{\partial x_i \partial x_j} h \Big|_{\boldsymbol{x}=\boldsymbol{x}_0} := \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} h \Big|_{\boldsymbol{x}=\boldsymbol{x}_0}$. For an arbitrary function $h : \mathbb{R}^{m_x + m_y} \to \mathbb{R}$, define $\nabla_{\boldsymbol{x}} h(\boldsymbol{x}, \boldsymbol{y})$ as the gradient of h with respect to $\boldsymbol{x}$. The sub-level and super-level sets of h are given by $\{\boldsymbol{x} \in \mathbb{R}^m : h(\boldsymbol{x}) \leq \eta\}$ and $\{\boldsymbol{x} \in \mathbb{R}^m : h(\boldsymbol{x}) \geq \eta\}$, respectively, as η varies over $\mathbb{R}$. Let $\mathbf{0}$ denote a vector of zeros of appropriate dimension.

Also, define

$$\mathcal{P}' := \{\boldsymbol{r} \in \mathbb{R}^{|\mathcal{M}|} \ : \ \boldsymbol{A}\boldsymbol{r} \le \boldsymbol{b},\ \mathbf{1}^\top \boldsymbol{r} = 0\},$$

and hence, $\boldsymbol{r} \in \mathcal{P}$ and $\mathbf{1}^\top \boldsymbol{r} = 0$ is succinctly represented as $\boldsymbol{r} \in \mathcal{P}'$, henceforth.

Our proofs shall make use of two known results in the literature. We describe them briefly. Consider a game $\mathcal{G}$ with (a) players $1, \ldots, N$, (b) actions $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$ for $i = 1, \ldots, N$, where $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathcal{X} \subseteq \mathbb{R}^{n_1 + \ldots + n_N}$, and (c) payoff functions $\varphi_i : \mathbb{R}^{n_1+\ldots+n_N} \to \mathbb{R}$ for each player $i = 1, \ldots N$.

- Game $\mathcal{G}$ is said to be *concave*, if $\mathcal{X}$ is a compact convex set, and φ_i is concave in $\boldsymbol{x}_i$ for each $i = 1, \ldots, N$. Then, Theorem 1 in [Ros65] states that a Nash equilibrium always exists for a concave game.

- Recall that game $\mathcal{G}$ is said to be a *weighted potential game*, if there exists a vector of weights $w \in \mathbb{R}^N_{++}$ and a potential function $\Phi : \mathbb{R}^{n_1+\ldots+n_N} \to \mathbb{R}$ that satisfies (5). Then, Theorem 1 in [Ney97] implies that if $\mathcal{G}$ is a weighted potential game with Φ as the potential function, and Φ is concave and continuously differentiable, then, any Nash equilibrium of $\mathcal{G}$ is an optimizer of Φ over $\mathcal{X}$.[5]

A.1 Proof of Theorem 1

From the definition of $\hat{\Pi}$, it follows that $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - \Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ does not depend on $\boldsymbol{r}$. Hence, for every $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}$, we have

$$\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - \hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}'; \boldsymbol{\theta}) = \Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - \Pi(\boldsymbol{q}, \boldsymbol{r}'; \boldsymbol{\theta}) \tag{15}$$

for each $\boldsymbol{r}, \boldsymbol{r}' \in \mathcal{P}'$. Expanding $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$, we obtain

$$\begin{aligned}\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) = (\theta_M + \theta_P - \theta_C) \sum_{m \in \mathcal{M}} \left[\sum_{f \in \mathcal{F}(m)} \left((\alpha_m - \beta_m r_m)\, q_f - c_f(q_f) \right) - \frac{\beta_m}{2} \sum_{f, f' \in \mathcal{F}(m)} q_f q_{f'} \right] \\ + (\theta_C - 2\theta_M) \sum_{m \in \mathcal{M}} \frac{\beta_m}{2} r_m^2 + \theta_M \sum_{m \in \mathcal{M}} \alpha_m r_m.\end{aligned}$$

[5] Neyman's result guarantees that if a game with finitely many players admits a concave and continuously differentiable potential function, then, any correlated equilibrium of the game is a pure strategy Nash equilibrium, and is given by a global optimizer of the potential function of the game.

Then, for every $f \in \mathcal{F}$ and $\boldsymbol{r} \in \mathcal{P}'$, it follows that

$$\begin{aligned}
&\hat{\Pi}(q_f, \boldsymbol{q}_{-f}, \boldsymbol{r}; \boldsymbol{\theta}) - \hat{\Pi}(q'_f, \boldsymbol{q}_{-f}, \boldsymbol{r}; \boldsymbol{\theta}) \\
&= (\theta_M + \theta_P - \theta_C) \left[\left(\alpha_{\mathcal{M}(f)} - \beta_{\mathcal{M}(f)} r_{\mathcal{M}(f)}\right) \left(q_f - q'_f\right) - \beta_{\mathcal{M}(f)} \left(\sum_{f' \in \mathcal{F}(\mathcal{M}(f))} q_{f'} \right) q_f \right. \\
&\qquad\qquad \left. + \beta_{\mathcal{M}(f)} \left(q'_f + \sum_{f' \in \mathcal{F}(\mathcal{M}(f)) \setminus \{f\}} q_{f'} \right) q'_f - \left(c_f(q_f) - c_f(q'_f)\right) \right] \\
&= (\theta_M + \theta_P - \theta_C) \left[\left(\alpha_{\mathcal{M}(f)} - \beta_{\mathcal{M}(f)} \left(r_{\mathcal{M}(f)} + \sum_{f' \in \mathcal{F}(\mathcal{M}(f))} q_{f'} \right) \right) q_f - c_f(q_f) \right. \\
&\qquad\qquad \left. - \left(\alpha_{\mathcal{M}(f)} - \beta_{\mathcal{M}(f)} \left(r_{\mathcal{M}(f)} + q'_f + \sum_{f' \in \mathcal{F}(\mathcal{M}(f)) \setminus \{f\}} q_{f'} \right) \right) q'_f + c_f(q'_f) \right] \\
&= (\theta_M + \theta_P - \theta_C) \left[\pi_f(q_f, \boldsymbol{q}_{-f}, \boldsymbol{r}) - \pi_f(q'_f, \boldsymbol{q}_{-f}, \boldsymbol{r}) \right]
\end{aligned}$$

for each $q_f, q'_f \in \mathbb{R}_+$. Hence, if $\theta_M + \theta_P - \theta_C > 0$, then, $\mathcal{G}(\boldsymbol{\theta})$ is a weighted potential game with $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ as the potential function.

A.2 Proof of Theorem 2(a)

We prove the existence of a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ for $2\theta_M - \theta_C \geq 0$ and $\theta_M + \theta_P - \theta_C > 0$ separately.

When $2\theta_M - \theta_C \geq 0$. We leverage Rosen's result to show that a Nash equilibrium exists. Note that $\mathcal{G}(\boldsymbol{\theta})$ is not a concave game since its joint strategy set is unbounded. Our key idea is to define another game $\hat{\mathcal{G}}(\boldsymbol{\theta})$ with a bounded joint strategy set such that any Nash equilibrium of $\hat{\mathcal{G}}(\boldsymbol{\theta})$ is also a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$. Then, we utilize Rosen's result to guarantee the existence of a Nash equilibrium of $\hat{\mathcal{G}}(\boldsymbol{\theta})$, and by extension, to that of $\mathcal{G}(\boldsymbol{\theta})$.

Recall that $\boldsymbol{r} \in \mathcal{P}'$. Since $\mathcal{P}'$ is compact, there exists $\bar{r} \in \mathbb{R}_+$ such that $|r_m| \leq \bar{r}$ for every $m \in \mathcal{M}$. Let $\bar{q} := \max_{f \in \mathcal{F}} (1/2) \left(\alpha_{\mathcal{M}(f)}/\beta_{\mathcal{M}(f)} + \bar{r}\right)$. Consider a game $\hat{\mathcal{G}}(\boldsymbol{\theta})$ that is identical to $\mathcal{G}(\boldsymbol{\theta})$ except that each firm $f \in \mathcal{F}$ has a strategy set $[0, \bar{q}]$. Then, $(\boldsymbol{q}, \boldsymbol{r}) \in [0, \bar{q}]^{|\mathcal{F}|} \times \mathcal{P}'$ is a Nash equilibrium of $\hat{\mathcal{G}}(\boldsymbol{\theta})$ if

$$\begin{aligned}
\pi_f(\boldsymbol{q}, \boldsymbol{r}) &\geq \pi_f(q'_f, \boldsymbol{q}_{-f}, \boldsymbol{r}), && \text{for all } q'_f \in [0, \bar{q}]; \\
\Pi(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) &\geq \Pi(\boldsymbol{q}, \boldsymbol{r}'; \boldsymbol{\theta}), && \text{for all } \boldsymbol{r}' \in \mathcal{P}'.
\end{aligned}$$

Now, π_f is continuous in its arguments. It is also concave in q_f because

$$\frac{\partial^2}{\partial q_f^2}\pi_f(\boldsymbol{q},\boldsymbol{r}) = -2\beta_{\mathcal{M}(f)} - c_f''(q_f) < 0.$$

Moreover, we have

$$\begin{aligned}\frac{\partial}{\partial q_f}\pi_f(\boldsymbol{q},\boldsymbol{r})\bigg|_{q_f=\bar{q}} &= \alpha_{\mathcal{M}(f)} - \beta_{\mathcal{M}(f)}r_{\mathcal{M}(f)} - 2\beta_{\mathcal{M}(f)}\bar{q} - c_f'(\bar{q}) \\ &\leq \alpha_{\mathcal{M}(f)} + \beta_{\mathcal{M}(f)}\bar{r} - 2\beta_{\mathcal{M}(f)}\bar{q} \\ &\leq 0.\end{aligned}$$

The first inequality holds since c_f is nondecreasing and $|r_{\mathcal{M}(f)}| \leq \bar{r}$. The second inequality follows from the definition of $\bar{q}$. Hence, we infer that $\pi_f(\boldsymbol{q},\boldsymbol{r})$ is decreasing over $q_f \geq \bar{q}$. In turn, it implies that if $(\boldsymbol{q},\boldsymbol{r})$ is a Nash equilibrium of $\hat{\mathcal{G}}(\boldsymbol{\theta})$, then,

$$\pi_f(\boldsymbol{q},\boldsymbol{r}) \geq \pi_f(q_f', \boldsymbol{q}_{-f}, \boldsymbol{r})$$

for all $q_f' \in \mathbb{R}_+$. Hence, any Nash equilibrium of $\hat{\mathcal{G}}(\boldsymbol{\theta})$ is also a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$.

Next, we argue that $\hat{\mathcal{G}}(\boldsymbol{\theta})$ is a concave game. The joint strategy set of $\hat{\mathcal{G}}(\boldsymbol{\theta})$ is given by the compact set $[0,\bar{q}]^{|\mathcal{F}|} \times \mathcal{P}'$. The payoff function of firm f, i.e., π_f, is continuous in all its arguments and concave in q_f. The market maker's payoff function $\Pi(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta})$ is continuous in all its arguments, and we have

$$\frac{\partial^2}{\partial r_m \partial r_{m'}}\Pi(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta}) = \begin{cases}-(2\theta_M - \theta_C)\beta_m, & \text{if } m = m', \\ 0, & \text{otherwise.}\end{cases} \tag{16}$$

Since $-(2\theta_M - \theta_C)\beta_m \leq 0$, the Hessian of $\Pi(\boldsymbol{q},\boldsymbol{r},\boldsymbol{\theta})$ with respect to $\boldsymbol{r}$ is then negative semidefinite, and hence, $\Pi(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta})$ is concave in $\boldsymbol{r}$. We conclude that $\hat{\mathcal{G}}(\boldsymbol{\theta})$ is a concave game, and therefore, has a Nash equilibrium. Hence, $\mathcal{G}(\boldsymbol{\theta})$ has a Nash equilibrium.

When $\theta_M + \theta_P - \theta_C > 0$. For such a design parameter, Theorem 1 implies that $\mathcal{G}(\boldsymbol{\theta})$ is a weighted potential game. Then, if $\mathcal{C}(\boldsymbol{\theta})$ has a finite optimizer, this optimizer is a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$, and hence, a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ exists. We show that the super-level sets of $\hat{\Pi}(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta})$ are compact, and hence, $\hat{\Pi}(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta})$ has a finite optimizer. Rewrite $\hat{\Pi}(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta})$ as

$$\hat{\Pi}(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta}) = -(\theta_M + \theta_P - \theta_C)h_1(\boldsymbol{q},\boldsymbol{r}) + h_2(\boldsymbol{r}),$$

where $h_1(\boldsymbol{q}, \boldsymbol{r})$ and $h_2(\boldsymbol{r})$ are defined as

$$h_1(\boldsymbol{q}, \boldsymbol{r}) := \sum_{m \in \mathcal{M}} \left[\frac{\beta_m}{2} \left(\left(\sum_{f \in \mathcal{F}(m)} q_f \right)^2 + \sum_{f \in \mathcal{F}(m)} q_f^2 \right) \right.$$

$$\left. - \sum_{f \in \mathcal{F}(m)} (\alpha_m q_f - c_f(q_f)) + \beta_m r_m \sum_{f \in \mathcal{F}(m)} q_f \right],$$

$$h_2(\boldsymbol{r}) := \left(\frac{\theta_C}{2} - \theta_M \right) \sum_{m \in \mathcal{M}} \beta_m r_m^2 + \sum_{m \in \mathcal{M}} (\theta_M \alpha_m + \theta_C \beta_m) r_m.$$

Recall that $\mathcal{P}'$ is compact, and hence, there exists $\bar{r} \in \mathbb{R}_+$ such that $|r_m| \leq \bar{r}$ for all $\boldsymbol{r} \in \mathcal{P}'$. It is easy to verify from the definition of h_1 that

$$h_1(\boldsymbol{q}, \boldsymbol{r}) \leq h_1(\boldsymbol{q}, \bar{r}\mathbf{1}),$$

for all $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}$. Continuity of h_1 and h_2 implies that the super-level set of $\hat{\Pi}$ are closed. Further, since c_f is convex and non-decreasing, we have that $\lim_{\|\boldsymbol{q}\| \to \infty} h_1(\boldsymbol{q}, \bar{r}\mathbf{1}) \to \infty$. Hence, its sub-level sets are bounded. In turn, it implies that the sub-level sets of $h_1(\boldsymbol{q}, \boldsymbol{r})$ are bounded. Also, $h_2(\boldsymbol{r})$ solely depends on $\boldsymbol{r}$, that varies over a compact set $\mathcal{P}'$. Then, h_2 takes values over a bounded set in $\mathbb{R}$. Putting the arguments together then implies that the super-level sets of $\hat{\Pi}$ are compact.

A.3 Proof of Theorem 2(b)

When $2\theta_M - \theta_C \geq \gamma \cdot (\theta_M + \theta_P - \theta_C) \geq 0$, part (a) implies that $\mathcal{G}(\boldsymbol{\theta})$ has at least one Nash equilibrium. To equate the set of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ to the set of optimizers of $\mathcal{C}(\boldsymbol{\theta})$, we first establish that the potential function $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ is jointly concave in $(\boldsymbol{q}, \boldsymbol{r})$. The Hessian of $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ with respect to $(\boldsymbol{q}, \boldsymbol{r})$ can be shown to be block diagonal with $|\mathcal{M}|$ block matrices. Denote said block matrices by $\boldsymbol{H}_m, m \in \mathcal{M}$, where

$$\boldsymbol{H}_m = - \begin{pmatrix} (\theta_M + \theta_P - \theta_C) \left(\beta_m \mathbf{1}\mathbf{1}^\top + \mathrm{diag}(\boldsymbol{d}_m) \right) & (\theta_M + \theta_P - \theta_C) \, \beta_m \mathbf{1} \\ (\theta_M + \theta_P - \theta_C) \, \beta_m \mathbf{1}^\top & (2\theta_M - \theta_C) \, \beta_m \end{pmatrix},$$

and $\boldsymbol{d}_m := \left(\beta_m + c_f''(q_f), f \in \mathcal{F}(m) \right) \in \mathbb{R}_+^{|\mathcal{F}(m)|}$. It suffices to show that $\boldsymbol{H}_m \preceq 0$ for each $m \in \mathcal{M}$. By the Sherman-Woodbury matrix identity (see [HJ12] for example), we have

$$\left(\beta_m \mathbf{1}\mathbf{1}^\top + \mathrm{diag}(\boldsymbol{d}_m) \right)^{-1}$$
$$= \mathrm{diag}\left(\boldsymbol{d}_m\right)^{-1} - \left(\frac{1}{1/\beta_m + \mathbf{1}^\top \mathrm{diag}\left(\boldsymbol{d}_m\right)^{-1} \mathbf{1}} \right) \mathrm{diag}\left(\boldsymbol{d}_m\right)^{-1} \mathbf{1}\mathbf{1}^\top \mathrm{diag}\left(\boldsymbol{d}_m\right)^{-1}.$$

Using Schur complements, we obtain

$$\begin{aligned}\boldsymbol{H}_m \preceq 0 &\iff (2\theta_M - \theta_C)\beta_m - (\theta_M + \theta_P - \theta_C)\beta_m \mathbf{1}^\top (\beta_m \mathbf{1}\mathbf{1}^\top + \mathrm{diag}(\boldsymbol{d}_m))^{-1}\beta_m \mathbf{1} \geq 0 \\ &\iff (2\theta_M - \theta_C) - (\theta_M + \theta_P - \theta_C)\left(1 - \frac{1}{1 + \beta_m \mathbf{1}^\top \mathrm{diag}\left(\boldsymbol{d}_m\right)^{-1}\mathbf{1}}\right) \geq 0. \end{aligned} \tag{17}$$

And, $\boldsymbol{H}_m \prec 0$ follows from the observation that

$$1 - \frac{1}{1 + \beta_m \mathbf{1}^\top \mathrm{diag}\left(\boldsymbol{d}_m\right)^{-1}\mathbf{1}} \leq 1 - \left(1 + \sum_{f \in \mathcal{F}(m)} \frac{\beta_m}{\beta_m + \inf_{q_f \geq 0} c''_f(q_f)}\right)^{-1} \leq \gamma.$$

Also, $\hat{\Pi}\left(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}\right)$ is continuously differentiable. Thus, $\mathcal{G}(\boldsymbol{\theta})$ is a potential game with a concave and continuously differentiable potential function. Neyman's result then implies that the set of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ is identical to the set of optimizers of $\mathcal{C}(\boldsymbol{\theta})$.

When $2\theta_M - \theta_C > \gamma \cdot (\theta_M + \theta_P - \theta_C) \geq 0$, it follows from (17) that $\hat{\Pi}\left(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}\right)$ is strictly concave. Then, $\mathcal{C}(\boldsymbol{\theta})$ has at most one optimizer. Finally, the desired result follows from the equality of the sets of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ and the optimizers of $\mathcal{C}(\boldsymbol{\theta})$, and the fact that $\mathcal{G}(\boldsymbol{\theta})$ has at least one Nash equilibrium.

A.4 Proof of Proposition 1

Since $\theta_M + \theta_P - \theta_C > 0$ and $2\theta_M - \theta_C > \gamma \cdot (\theta_M + \theta_P - \theta_C)$, we infer from Theorem 1 that $\mathcal{G}(\boldsymbol{\theta})$ is a weighted potential game with a strictly concave and continuously differentiable potential function $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$. Hence, the set of Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ is identical to the set of optimizers of $\mathcal{C}(\boldsymbol{\theta})$. Moreover, since $\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ is strictly concave, $\mathcal{C}(\boldsymbol{\theta})$ has at most one finite optimizer. When $\mathcal{P} = \mathbb{R}^{|\mathcal{M}|}$, then, $\mathcal{C}(\boldsymbol{\theta})$ seeks to maximize $\hat{\Pi}\left(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}\right)$ subject to $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}$ and $\mathbf{1}^\top \boldsymbol{r} = 0$. Now, $\mathcal{C}(\boldsymbol{\theta})$ being equivalent to a convex optimization problem with linear constraints, the KKT conditions imply that $(\boldsymbol{q}, \boldsymbol{r})$ solves $\mathcal{C}(\boldsymbol{\theta})$ if and only if $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}$, $\mathbf{1}^\top \boldsymbol{r} = 0$, and there exists $\boldsymbol{\mu} \in \mathbb{R}_+^{|\mathcal{F}|}$ and $\lambda \in \mathbb{R}$ that satisfy

$$\nabla_{\boldsymbol{r}} \left[\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - \lambda \mathbf{1}^\top \boldsymbol{r}\right] = \mathbf{0}, \tag{18a}$$

$$\nabla_{\boldsymbol{q}} \left[\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) + \boldsymbol{\mu}^\top \boldsymbol{q}\right] = \mathbf{0}, \tag{18b}$$

$$\boldsymbol{\mu}^\top \boldsymbol{q} = 0, \tag{18c}$$

Let $\boldsymbol{\mu} \in \mathbb{R}_+^{|\mathcal{F}|}$ be the all-zero vector, and

$$\lambda := \frac{1}{2}\left(\theta_P + \theta_M - \theta_C\right)\left[\frac{1}{|\mathcal{F}|}\sum_{f'} C_{f'} - \alpha\right] + \theta_M \alpha. \tag{19}$$

In what follows, we show that $\boldsymbol{q}, \boldsymbol{r}$ defined in (10) – (11), and $\boldsymbol{\mu}, \lambda$ defined above, together satisfy the Karush-Kuhn-Tucker optimality conditions.

Using the lower bound on α, we infer that $\boldsymbol{q} \in \mathbb{R}_+^{|\mathcal{F}|}$. Also, it is easy to verify that $\mathbf{1}^\top \boldsymbol{r} = 0$. Substituting the values of $\boldsymbol{q}, \boldsymbol{r}$ into the left hand side of (18a), we get

$$\begin{aligned}\frac{\partial}{\partial r_{\mathcal{M}(f)}} \left[\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) - \lambda \mathbf{1}^\top \boldsymbol{r}\right] &= -\left(2\theta_M - \theta_C\right) \beta r_{\mathcal{M}(f)} - \left(\theta_P + \theta_M - \theta_C\right) \beta q_f + \theta_M \alpha - \lambda \\ &= 0,\end{aligned}$$

where the last step follows from (19). Similarly, substituting the values of $\boldsymbol{q}, \boldsymbol{r}$ into the left hand side of (18b) gives

$$\frac{\partial}{\partial q_f} \left[\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta}) + \boldsymbol{\mu}^\top \boldsymbol{q}\right] = \alpha - \beta \left(r_{\mathcal{M}(f)} + 2q_f\right) - C_f + \mu_f = 0.$$

Finally, (18c) trivially holds, since $\boldsymbol{\mu}$ is the all-zero vector. Hence, we conclude that $(\boldsymbol{q}, \boldsymbol{r})$, as defined in (10) – (11), defines the unique Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$.

Towards computing the social welfare at the unique Nash equilibrium, we obtain

$$\begin{aligned}\sum_{m \in \mathcal{M}} \mathsf{CS}_m(\boldsymbol{q}, \boldsymbol{r}) &= \frac{\beta}{2} \sum_{f \in \mathcal{F}} \left(q_f + r_{\mathcal{M}(f)}\right)^2 \\ &= \frac{1}{8\beta} \left[|\mathcal{F}|(\alpha - \bar{C})^2 + (\kappa(\boldsymbol{\theta}) - 1)^2 \sum_{f \in \mathcal{F}} (C_f - \bar{C})^2 \right]; \\ \sum_{m \in \mathcal{M}} \mathsf{PS}_m(\boldsymbol{q}, \boldsymbol{r}) &= \sum_{f \in \mathcal{F}} \left[\alpha - C_f - \beta \left(q_f + r_{\mathcal{M}(f)}\right)\right] \cdot q_f \\ &= \frac{1}{4\beta} \left[|\mathcal{F}|(\alpha - \bar{C})^2 + (\kappa(\boldsymbol{\theta}) + 1)^2 \sum_{f \in \mathcal{F}} (C_f - \bar{C})^2 \right]; \\ \sum_{m \in \mathcal{M}} \mathsf{MS}_m(\boldsymbol{q}, \boldsymbol{r}) &= -\beta \sum_{f \in \mathcal{F}} \left(q_f + r_{\mathcal{M}(f)}\right) \cdot r_{\mathcal{M}(f)} \\ &= -\frac{1}{2\beta} \kappa(\boldsymbol{\theta})(\kappa(\boldsymbol{\theta}) - 1) \sum_{f \in \mathcal{F}} (C_f - \bar{C})^2.\end{aligned}$$

The details are omitted. Then, the social welfare at the unique Nash equilibrium is given by

$$\begin{aligned}
&\sum_{m\in\mathcal{M}} [\mathsf{CS}_m(\boldsymbol{q},\boldsymbol{r}) + \mathsf{PS}_m(\boldsymbol{q},\boldsymbol{r}) + \mathsf{MS}_m(\boldsymbol{q},\boldsymbol{r})] \\
&= \frac{3|\mathcal{F}|}{8\beta}\left(\alpha-\bar{C}\right)^2 + \frac{1}{8\beta}\left[(\kappa(\boldsymbol{\theta})-1)^2 + 2(\kappa(\boldsymbol{\theta})+1)^2 - 4\kappa(\boldsymbol{\theta})(\kappa(\boldsymbol{\theta})-1)\right]\sum_{f\in\mathcal{F}}\left(C_f-\bar{C}\right)^2 \\
&= \frac{3|\mathcal{F}|}{8\beta}\left(\alpha-\bar{C}\right)^2 + \frac{1}{8\beta}\left(-\kappa(\boldsymbol{\theta})^2 + 6\kappa(\boldsymbol{\theta}) + 3\right)\sum_{f\in\mathcal{F}}\left(C_f-\bar{C}\right)^2.
\end{aligned}$$

Finally, utilizing the definition of σ_c^2 in the above equation yields the desired result.

A.5 Proof of Lemma 1

$\mathcal{C}(\boldsymbol{\theta})$ is a multiparametric convex program, parameterized by $\boldsymbol{\theta}$. When $\boldsymbol{\theta}\in\Theta_\varepsilon$, $\mathcal{C}(\boldsymbol{\theta})$ admits a unique primal optimal solution for each $\boldsymbol{\theta}\in\Theta_\varepsilon$; call it $(\boldsymbol{q}(\boldsymbol{\theta}),\boldsymbol{r}(\boldsymbol{\theta}))$. Let $(\lambda(\boldsymbol{\theta}),\mu(\boldsymbol{\theta}))$ denote a dual optimal solution of $\mathcal{C}(\boldsymbol{\theta})$. Dual optimal solutions may not be unique. Since Θ_ε is compact, it suffices to argue that there exists positive constants $\bar{q}$, $\bar{r}$, $\bar{\lambda}$, and $\bar{\mu}$, such that $\mathcal{C}(\boldsymbol{\theta})$ admits a primal/dual pair of optimal solutions that satisfies

$$\|\boldsymbol{q}(\boldsymbol{\theta})\|_2\le\bar{q},\ \|\boldsymbol{r}(\boldsymbol{\theta})\|_2\le\bar{r},\ \|\boldsymbol{\lambda}(\boldsymbol{\theta})\|_2\le\bar{\lambda},\ \|\boldsymbol{\mu}(\boldsymbol{\theta})\|_2\le\bar{\mu}$$

for each $\boldsymbol{\theta}\in\Theta_\varepsilon$.

If $\boldsymbol{r}$ is feasible in $\mathcal{C}(\boldsymbol{\theta})$, then $\boldsymbol{r}\in\mathcal{P}'$, where $\mathcal{P}'$ is compact. That yields a uniform bound on $\|\boldsymbol{r}(\boldsymbol{\theta})\|_2$ for all $\boldsymbol{\theta}\in\Theta_\varepsilon$. Call that bound $\bar{r}$.

Recall that $(\boldsymbol{q}(\boldsymbol{\theta}),\boldsymbol{r}(\boldsymbol{\theta}))$ is also the unique Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$. Therefore, $q_f(\boldsymbol{\theta})$ maximizes $\pi_f\left(q_f,\boldsymbol{q}_{-f}(\boldsymbol{\theta}),\boldsymbol{r}(\boldsymbol{\theta})\right)$ over $q_f\ge 0$. However, we also have

$$\begin{aligned}
\frac{\partial}{\partial q_f}\pi_f\left(q_f,\boldsymbol{q}_{-f}(\boldsymbol{\theta}),\boldsymbol{r}(\boldsymbol{\theta})\right) &= \alpha_{\mathcal{M}(f)} - \beta_{\mathcal{M}(f)} r_{\mathcal{M}(f)}(\boldsymbol{\theta}) - 2\beta_{\mathcal{M}(f)} q_f - c_f'(q_f) \\
&\le \alpha_{\mathcal{M}(f)} + \beta_{\mathcal{M}(f)}\bar{r} - 2\beta_{\mathcal{M}(f)} q_f,
\end{aligned}$$

that implies $\pi_f\left(q_f,\boldsymbol{q}_{-f}(\boldsymbol{\theta}),\boldsymbol{r}(\boldsymbol{\theta})\right)$ decreases with q_f for $q_f > \frac{\bar{r}}{2} + \max_{m\in\mathcal{M}}\frac{\alpha_m}{2\beta_m}$. In turn, we conclude

$$\|\boldsymbol{q}(\boldsymbol{\theta})\|_2 \le \bar{q} := |\mathcal{F}|\left(\frac{\bar{r}}{2} + \max_{m\in\mathcal{M}}\frac{\alpha_m}{2\beta_m}\right).$$

For each $f\in\mathcal{F}$, we have assumed c_f to be continuously differentiable, implying $\nabla_{\boldsymbol{q}}\hat{\Pi}(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta})$ is continuous in $(\boldsymbol{q},\boldsymbol{r},\boldsymbol{\theta})$. Therefore, $\nabla_{\boldsymbol{q}}\hat{\Pi}(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta})$ remains bounded over $\{\boldsymbol{q}:\|\boldsymbol{q}\|_2\le\bar{q}\}\times\mathcal{P}'\times\Theta_\varepsilon$. As a result, $\mu(\boldsymbol{\theta}) = -\left.\nabla_{\boldsymbol{q}}\hat{\Pi}(\boldsymbol{q},\boldsymbol{r};\boldsymbol{\theta})\right|_{\boldsymbol{q}(\boldsymbol{\theta}),\boldsymbol{r}(\boldsymbol{\theta})}$ admits a uniform bound $\bar{\mu}$.

Let $\Lambda(\boldsymbol{\theta})$ denote the set of optimal Lagrange multipliers for the constraint $\boldsymbol{A}'^\top\boldsymbol{r}\le\boldsymbol{b}'$ in $\mathcal{C}(\boldsymbol{\theta})$.

We conclude the proof by showing that $\inf_{\boldsymbol{\lambda}(\boldsymbol{\theta})\in\Lambda(\boldsymbol{\theta})} \|\boldsymbol{\lambda}(\boldsymbol{\theta})\|_2$ is uniformly bounded over $\boldsymbol{\theta} \in \Theta_\varepsilon$. To that end, suppose the rows of $\boldsymbol{A}'$ are given by $\boldsymbol{a}_1^\top, \ldots, \boldsymbol{a}_{\dim(\boldsymbol{b}')}^\top$, where $\boldsymbol{a}_i \in \mathbb{R}^{|\mathcal{M}|}$. Denote by $\mathcal{S}(\boldsymbol{\theta})$, the set of active constraints at optimality of $\mathcal{C}(\boldsymbol{\theta})$. That is, $\boldsymbol{a}_i^\top \boldsymbol{r}(\boldsymbol{\theta}) = \boldsymbol{b}_i'$ for each $i \in \mathcal{S}(\boldsymbol{\theta}) \subseteq \{1, \ldots, \dim(\boldsymbol{b}')\}$, and $\boldsymbol{a}_i^\top \boldsymbol{r}(\boldsymbol{\theta}) < \boldsymbol{b}_i'$ for $i \in \mathcal{S}^c(\boldsymbol{\theta}) := \{1, \ldots, \dim(\boldsymbol{b}')\} \setminus \mathcal{S}(\boldsymbol{\theta})$. Then, $\Lambda(\boldsymbol{\theta})$ is given by

$$\Lambda(\boldsymbol{\theta}) = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^{\dim(\boldsymbol{b}')} : \lambda_i = 0 \text{ for } i \in \mathcal{S}^c(\boldsymbol{\theta}), \sum_{i\in\mathcal{S}(\boldsymbol{\theta})} \lambda_i a_i = -\nabla_{\boldsymbol{r}} \hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})\Big|_{\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta})} \right\}. \tag{20}$$

If $\boldsymbol{A}'(\boldsymbol{\theta})$ denotes the $|\mathcal{S}(\boldsymbol{\theta})| \times |\mathcal{M}|$ matrix with rows $\boldsymbol{a}_i^\top$ for $i \in \mathcal{S}(\boldsymbol{\theta})$, we conclude from (20) that

$$\inf_{\boldsymbol{\lambda}(\boldsymbol{\theta})\in\Lambda(\boldsymbol{\theta})} \|\boldsymbol{\lambda}(\boldsymbol{\theta})\|_2 = \left\| [\boldsymbol{A}'^\top(\boldsymbol{\theta})]^\dagger \nabla_{\boldsymbol{r}} \hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})\Big|_{\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{r}(\boldsymbol{\theta})} \right\|_2,$$

where $[\boldsymbol{A}'^\top(\boldsymbol{\theta})]^\dagger$ denotes the Moore-Penrose inverse of $\boldsymbol{A}'^\top(\boldsymbol{\theta})$. Using the continuous differentiability of $c_f, f \in \mathcal{F}$, one can argue that $\nabla_{\boldsymbol{r}}\hat{\Pi}(\boldsymbol{q}, \boldsymbol{r}; \boldsymbol{\theta})$ remains bounded over $\{\boldsymbol{q} : \|\boldsymbol{q}\|_2 \leq \bar{q}\} \times \mathcal{P}' \times \Theta_\varepsilon$. And, the rest follows from the fact that $\boldsymbol{A}'(\boldsymbol{\theta})$ has finitely many possibilities for $\boldsymbol{\theta} \in \Theta_\varepsilon$.

B Analyzing the Two-Market Two-Firm Example in Fig. 1

This section is devoted to deriving all Nash equilibria of $\mathcal{G}(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \mathbb{R}_+^3$ in a two-market two-firm example, portrayed in Figure 1. Our formulae let us gain insights into the parametric dependence of the Nash equilibria on the design parameter.

Consider $\mathcal{G}(\boldsymbol{\theta})$, where $\mathcal{M} = \{1, 2\}$, $\mathcal{F} = \{1, 2\}$, and $\mathcal{F}(1) = \{1\}$, $\mathcal{F}(2) = \{2\}$. Each firm has an increasing linear cost, given by $c_f(q_f) = C_f q_f$. Assume that the markets are spatially homogeneous, having inverse linear demand functions $p_m(d_m) = \alpha - \beta d_m$ for $m \in \mathcal{M}$, where $\alpha, \beta > 0$. The network constraint is given by $\mathcal{P} := \{\boldsymbol{r} = (r_1, r_2)^\top : |r_1| \leq b, \ |r_2| \leq b\}$, where b denotes the capacity of the link between the two markets. Hence, the markets only differ in the marginal costs of the firms supplying in each market.

Notice that $\mathbf{1}^\top \boldsymbol{r} = 0$ for our example implies $r_1 = -r_2$. Defining $r := r_1 = -r_2$, the market maker's strategy set can be described by $\{r \in \mathbb{R} : |r| \leq b\}$. And, the market maker's payoff (with a slight abuse of notation) is given by

$$\begin{aligned}\Pi(q_1, q_2, r; \boldsymbol{\theta}) = &-(\theta_M + \theta_P - \theta_C)(q_1 - q_2)\beta r - (2\theta_M - \theta_C)\beta r^2 \\ &+ \theta_P((\alpha - C_1) q_1 + (\alpha - C_2) q_2) + \frac{1}{2}(\theta_C - 2\theta_P)\beta\left(q_1^2 + q_2^2\right).\end{aligned}$$

Restrict attention to the case where

$$0 \leq b \leq \frac{\alpha - \max\{C_1, C_2\}}{\beta}.$$

Then, (q_1, q_2, r), where $q_1, q_2 \geq 0$ and $|r| \leq b$, constitutes a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ if and only if

1. $\Pi(q_1, q_2, r; \boldsymbol{\theta}) \geq \Pi(q_1, q_2, r'; \boldsymbol{\theta})$ for any r' such that $|r'| \leq b$, and

2. the production quantities satisfy

$$q_1 = \frac{1}{2}\left(\frac{\alpha - C_1}{\beta} - r\right), \quad q_2 = \frac{1}{2}\left(\frac{\alpha - C_2}{\beta} + r\right). \tag{21}$$

Let $\mathcal{R}(\boldsymbol{\theta})$ denote the set of all r's that comprise a Nash equilibrium for the game, when the design parameter is $\boldsymbol{\theta}$. Then, for each $r \in \mathcal{R}(\boldsymbol{\theta})$, the production quantities at the Nash equilibrium are uniquely identified by (21). We provide $\mathcal{R}(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \mathbb{R}^3_+$ in Table 1 by summarizing the results of Lemmas 2 – 4. We state and prove Lemmas 2 – 4 at the end of this section. Presenting our results require the following additional notation. For any $x \in \mathbb{R}$, define

$$[x]_\ell^u := \begin{cases} x, & \text{if } \ell \leq x \leq u, \\ \ell, & \text{if } x < \ell, \\ u, & \text{otherwise.} \end{cases}.$$

Let sgn (x) denote the sign of $x \in \mathbb{R}$. We denote the null set by $\varnothing$. For convenience, define

$$\Delta C := C_1 - C_2, \quad \text{and} \quad \kappa(\boldsymbol{\theta}) := \frac{\theta_P + \theta_M - \theta_C}{3\theta_M - \theta_C - \theta_P}.$$

Recall that Theorems 1 and 2 provide sufficient conditions on $\boldsymbol{\theta}$ for $\mathcal{G}(\boldsymbol{\theta})$ to exhibit certain properties. Though we do not address the question of necessity or tightness, we use the results in Table 1 for the two-market two-firm example to illustrate that each of the properties may fail to hold if the respective conditions are not satisfied.

1. When neither $\theta_M + \theta_P - \theta_C > 0$ nor $2\theta_M \geq \theta_C$ holds, a Nash equilibrium may not exist. Consider $\boldsymbol{\theta}$ that satisfies the above conditions, such that $\theta_M + \theta_P - \theta_C < 0$, and let $\left|\frac{\Delta C}{2\beta}\right| < b$ in our example. Then, Table 1 implies that $\mathcal{R}(\boldsymbol{\theta}) = \varnothing$.

2. When $2\theta_M \geq \theta_C$, but not $\theta_M + \theta_P - \theta_C > 0$, a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ exists, but $\mathcal{G}(\boldsymbol{\theta})$ is not a weighted potential game. Consider our example, where $2\theta_M - \theta_C = 0$, $\theta_M + \theta_P - \theta_C < 0$ and $C_1 = C_2$. From Table 1, the game admits a unique Nash equilibrium with $\mathcal{R}(\boldsymbol{\theta}) = \{0\}$. However, it can be shown that the actions of the players under best response dynamics exhibit

Conditions on $\boldsymbol{\theta}$		$\mathcal{R}(\boldsymbol{\theta})$
$2\theta_M - \theta_C > 0$	$3\theta_M - \theta_C - \theta_P > 0$	$\left\{ \left[\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right]_{-b}^{+b} \right\}$
	$3\theta_M - \theta_C - \theta_P = 0$	$\begin{cases} [-b, +b], & \text{if } \Delta C = 0, \\ \{b \cdot \text{sgn}(\Delta C)\}, & \text{otherwise.} \end{cases}$
	$3\theta_M - \theta_C - \theta_P < 0$	$\begin{cases} \left\{ \pm b, \dfrac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right\}, & \text{if } \left\| \dfrac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right\| \leq b, \\ \{b \cdot \text{sgn}(\Delta C)\}, & \text{otherwise.} \end{cases}$
$2\theta_M - \theta_C = 0$	$\theta_P + \theta_M - \theta_C < 0$	$\left\{ \left[-\frac{\Delta C}{2\beta} \right]_{-b}^{+b} \right\}$
	$\theta_P + \theta_M - \theta_C = 0$	$[-b, +b]$
	$\theta_P + \theta_M - \theta_C > 0$	$\begin{cases} \left\{ \pm b, -\dfrac{\Delta C}{2\beta} \right\}, & \text{if } \left\| \dfrac{\Delta C}{2\beta} \right\| \leq b, \\ \{b \cdot \text{sgn}(\Delta C)\}, & \text{otherwise.} \end{cases}$
$2\theta_M - \theta_C < 0$	$\theta_P + \theta_M - \theta_C < 0$	$\begin{cases} \{-b \cdot \text{sgn}(\Delta C)\}, & \text{if } \left\| \dfrac{\Delta C}{2\beta} \right\| \geq b, \\ \varnothing, & \text{otherwise.} \end{cases}$
	$\theta_P + \theta_M - \theta_C = 0$	$\{\pm b\}$
	$\theta_P + \theta_M - \theta_C > 0$	$\begin{cases} \{\pm b\}, & \text{if } \left\| \dfrac{\Delta C}{2\beta} \right\| \leq b, \\ \{b \cdot \text{sgn}(\Delta C)\}, & \text{otherwise.} \end{cases}$

Table 1: Summary of $\mathcal{R}(\boldsymbol{\theta})$ for the two-market two-firm example.

the following cycle:

$$\begin{array}{ccc} r & q_1 & q_2 \\ +b & \frac{1}{2}\left(\frac{\alpha - C_1}{\beta} - b\right) & \frac{1}{2}\left(\frac{\alpha - C_2}{\beta} + b\right) \\ -b & \frac{1}{2}\left(\frac{\alpha - C_1}{\beta} + b\right) & \frac{1}{2}\left(\frac{\alpha - C_2}{\beta} - b\right) \\ +b & \frac{1}{2}\left(\frac{\alpha - C_1}{\beta} - b\right) & \frac{1}{2}\left(\frac{\alpha - C_2}{\beta} + b\right) \\ \vdots & \vdots & \vdots \end{array}$$

implying that the game under consideration is not a potential game.

3. When $2\theta_M - \theta_C \geq 0$, and $\theta_M + \theta_P - \theta_C > 0$, but $2\theta_M - \theta_C \geq \gamma \cdot (\theta_M + \theta_P - \theta_C)$ does not hold, there may exist a Nash equilibrium of $\mathcal{G}(\boldsymbol{\theta})$ that is not an optimizer of $\mathcal{C}(\boldsymbol{\theta})$. In our example, $\gamma = \frac{1}{2}$, and hence,

$$2\theta_M - \theta_C - \gamma \cdot (\theta_M + \theta_P - \theta_C) = \frac{1}{2}(3\theta_M - \theta_C - \theta_P).$$

Consider $\boldsymbol{\theta}$ that satisfies $2\theta_M - \theta_C > 0$, $3\theta_M - \theta_C - \theta_P < 0$, and additionally, $\left| \frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right| < b$. Then, Table 1 reveals that $\mathcal{R}(\boldsymbol{\theta}) = \left\{ \pm b, \frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right\}$, and hence, the game has three distinct Nash equilibria.

Now, $\mathcal{C}(\boldsymbol{\theta})$ maximizes $\hat{\Pi}(q_1, q_2, r; \boldsymbol{\theta})$ over $q_1, q_2 \geq 0$ and $|r| \leq b$, where

$$\begin{aligned}\hat{\Pi}(q_1, q_2, r; \boldsymbol{\theta}) =& (\theta_M + \theta_P - \theta_C)\left((\alpha - C_1)q_1 + (\alpha - C_2)q_2 - \beta(q_1^2 + q_2^2) - (q_1 - q_2)\beta r\right) \\ &- (2\theta_M - \theta_C)\beta r^2.\end{aligned}$$

One can argue that if q_1, q_2, r solves $\mathcal{C}(\boldsymbol{\theta})$, then, q_1, q_2 are related to r as in equation (21). In turn, solving $\mathcal{C}(\boldsymbol{\theta})$ then reduces to maximizing

$$\frac{1}{2}(\theta_M + \theta_P - \theta_C)\left[r \cdot \Delta C + \frac{1}{2\beta}(\alpha - C_1)^2 + \frac{1}{2\beta}(\alpha - C_2)^2\right] - \frac{1}{2}\left(3\theta_M - \theta_C - \theta_P\right)\beta r^2,$$

subject to $|r| \leq b$. The above function being a strictly convex function in r attains its maximum at the boundary of the feasible set, i.e., at $\pm b$. As a result, there does *not* exist an optimizer of $\mathcal{C}(\boldsymbol{\theta})$ with $r = \frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}$.

4. When $2\theta_M - \theta_C = \gamma\left(\theta_M + \theta_P - \theta_C\right) > 0$, then $\mathcal{G}(\boldsymbol{\theta})$ may have a multitude of Nash equilibria, all of which are optimizers of $\mathcal{C}(\boldsymbol{\theta})$. This is observed in our example, where $\mathcal{R}(\boldsymbol{\theta}) = [-b, +b]$, when $2\theta_M - \theta_C > 0$, $3\theta_M - \theta_C - \theta_P = 0$, and $C_1 = C_2$.

In what follows, we formally characterize $\mathcal{R}(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \mathbb{R}^3_+$ for our example.

Lemma 2. *Suppose $\theta_P + \theta_M - \theta_C = 0$. Then, $\mathcal{R}(\boldsymbol{\theta})$ is given by*

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \{0\}, & \textit{if } 2\theta_M - \theta_C > 0, \\ \left[-b, +b\right], & \textit{if } 2\theta_M - \theta_C = 0, \\ \{\pm b\}, & \textit{otherwise}. \end{cases}$$

Proof. When $\theta_P + \theta_M - \theta_C = 0$, the maximizer of $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ over r is independent of q_1 and q_2. Further, if $2\theta_M - \theta_C > 0$, then $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is a concave quadratic even function of r, and hence, $r = 0$ is its unique maximizer. On the other hand, if $2\theta_M - \theta_C = 0$, then $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is independent of r, implying each $r \in [-b, b]$ constitutes a maximizer of $\Pi(q_1, q_2, r; \boldsymbol{\theta})$. Finally, if $2\theta_M - \theta_C < 0$, then $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is a convex quadratic even function of r, that attains its maximum at the boundaries of the feasible set, i.e., at $r = \pm b$. □

Lemma 3. *Suppose $\theta_P + \theta_M - \theta_C < 0$. Then, $\mathcal{R}(\boldsymbol{\theta})$ is given by*

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \left\{ \left[\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right]_{-b}^{+b} \right\}, & \text{if } 2\theta_M - \theta_C > 0, \\ \left\{ \left[-\frac{\Delta C}{2\beta} \right]_{-b}^{+b} \right\}, & \text{if } 2\theta_M - \theta_C = 0, \\ \{-b \cdot \text{sgn}\,(\Delta C)\}, & \text{if } 2\theta_M - \theta_C < 0, \text{ and } \left| \frac{\Delta C}{2\beta} \right| \geq b, \\ \varnothing, & \text{otherwise}. \end{cases}$$

Proof. Suppose $\theta_P + \theta_M - \theta_C < 0$. Define the expressions in (21) as $q_1(r)$ and $q_2(r)$, respectively, to make explicit its dependence on r. When $2\theta_M - \theta_C > 0$, the function $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is strictly concave in r, and hence, $q_1(r), q_2(r), r$ constitutes a Nash equilibrium of the game, if and only if one of three cases arise:

$$\rho(r, \boldsymbol{\theta}) = 0, \text{ and } |r| \leq b, \tag{22a}$$

$$\text{or } \rho(r, \boldsymbol{\theta}) \leq 0, \text{ and } r = -b, \tag{22b}$$

$$\text{or } \rho(r, \boldsymbol{\theta}) \geq 0, \text{ and } r = +b. \tag{22c}$$

where $\rho(r, \boldsymbol{\theta})$ is the derivative of $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ with respect to r, evaluated at $q_1(r), q_2(r), r$, given by

$$\rho(r, \boldsymbol{\theta}) = \frac{\Delta C}{2}(\theta_P + \theta_M - \theta_C) - (3\theta_M - \theta_C - \theta_P)\beta r. \tag{23}$$

Now, $\theta_P + \theta_M - \theta_C < 0$ and $2\theta_M - \theta_C > 0$ together imply $3\theta_M - \theta_C - \theta_P > 0$. Using the relations in (22a) – (22c), it is then straightforward to conclude that

$$\mathcal{R}(\boldsymbol{\theta}) = \left\{ \left[\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta} \right]_{-b}^{+b} \right\},$$

when $\theta_P + \theta_M - \theta_C < 0$ and $2\theta_M - \theta_C > 0$.

Next, consider the case, when $2\theta_M - \theta_C = 0$. Then, $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is a linear function of r with slope $\beta(q_1 - q_2)$. Thus, $q_1(r), q_2(r), r$ constitutes a Nash equilibrium of the game if and only if one of three cases arise: (i) $q_1(r) = q_2(r)$, and $|r| \leq b$, or (ii) $q_1(r) > q_2(r)$ and $r = +b$, or (iii) $q_1(r) < q_2(r)$ and $r = -b$. Substituting the values of $q_1(r)$ and $q_2(r)$ from (21), and rearranging, we get

$$\mathcal{R}(\boldsymbol{\theta}) = \left\{ \left[\frac{\Delta C}{2\beta} \right]_{-b}^{+b} \right\}.$$

Finally, consider the case when $2\theta_M - \theta_C < 0$. Then, $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is strictly convex in r, and is maximized at either $r = -b$ or $r = +b$ or both. Notice that

$$\Pi(q_1, q_2, +b; \boldsymbol{\theta}) - \Pi(q_1, q_2, -b; \boldsymbol{\theta}) = -2(\theta_M + \theta_P - \theta_C)(q_1 - q_2)\beta b,$$

implying (i) $+b$ is an optimizer if $q_1(+b) \geq q_2(+b)$, and (ii) $-b$ is an optimizer if $q_1(-b) \leq q_2(-b)$. Upon simplifying these conditions, we conclude,

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \{-b \cdot \text{sgn}\ (\Delta C)\}\,, & \text{if} \left|\frac{\Delta C}{2\beta}\right| \geq b, \\ \varnothing, & \text{otherwise.} \end{cases}$$

The rest follows from combining the values of $\mathcal{R}(\boldsymbol{\theta})$ under different cases. □

Lemma 4. *Suppose $\theta_P + \theta_M - \theta_C > 0$. Then, $\mathcal{R}(\boldsymbol{\theta})$ is given by*

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \left\{ \left[\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right]_{-b}^{+b} \right\}, & \text{if } 2\theta_M - \theta_C > 0, \text{ and } 3\theta_M - \theta_C - \theta_P > 0, \\ [-b, +b]\,, & \text{if } 2\theta_M - \theta_C > 0, 3\theta_M - \theta_C - \theta_P = 0, \text{ and } \Delta C = 0, \\ \left\{\pm b, \frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right\}, & \text{if } 2\theta_M - \theta_C > 0, 3\theta_M - \theta_C - \theta_P < 0, \text{ and } \left|\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right| \leq b, \\ \left\{\pm b, -\frac{\Delta C}{2\beta}\right\}, & \text{if } 2\theta_M - \theta_C = 0, \text{ and } \left|\frac{\Delta C}{2\beta}\right| \leq b, \\ \{\pm b\}\,, & \text{if } 2\theta_M - \theta_C < 0, \text{ and } \left|\frac{\Delta C}{2\beta}\right| \leq b, \\ \{b \cdot \text{sgn}\ (\Delta C)\}\,, & \text{otherwise.} \end{cases}$$

Proof. Suppose $\theta_P + \theta_M - \theta_C > 0$. First, consider the case, when $2\theta_M - \theta_C > 0$. Then, $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is strictly concave in r. Similar to our analysis in the proof of 3, it follows that $q_1(r), q_2(r), r$ constitute a Nash equilibrium of the game, if and only if one of three cases, given by (22a) – (22c), arises. We tackle three further cases separately, depending on the sign of $3\theta_M - \theta_C - \theta_P$.

Case (i): $3\theta_M - \theta_C - \theta_P > 0$. This case is identical to the case when $\theta_P + \theta_M - \theta_C < 0$ and $3\theta_M - \theta_C - \theta_P > 0$, and we obtain

$$\mathcal{R}(\boldsymbol{\theta}) = \left\{ \left[\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right]_{-b}^{+b} \right\}.$$

Case (ii): $3\theta_M - \theta_C - \theta_P = 0$. Since $\theta_P + \theta_M - \theta_C > 0$, the sign of $\rho(r, \boldsymbol{\theta})$, defined in (23), is given by the sign of ΔC. Then, (22a) – (22c) implies that

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} [-b, +b], & \text{if } \Delta C = 0, \\ \{b \cdot \text{sgn}\ (\Delta C), & \text{otherwise.} \end{cases}$$

Case (iii): $3\theta_M - \theta_C - \theta_P < 0$. We solve for r by setting $\rho(r, \boldsymbol{\theta}) = 0$, and further utilize (22a) – (22c) to obtain

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \left\{\pm b, \frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right\}, & \text{if } \left|\frac{\kappa(\boldsymbol{\theta})\Delta C}{2\beta}\right| \leq b, \\ \{b \cdot \text{sgn}\,(\Delta C)\}, & \text{otherwise.} \end{cases}$$

Second, consider that case when $2\theta_M - \theta_C = 0$. Then, $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is linear in r with slope $\beta(q_1 - q_2)$. The analysis is similar to the case when $2\theta_M - \theta_C = 0$, but with $\theta_P + \theta_M - \theta_C < 0$. Proceeding as in the proof of Lemma 3, we obtain

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \left\{\pm b, -\frac{\Delta C}{2\beta}\right\}, & \text{if} \left|\frac{\Delta C}{2\beta}\right| \leq b, \\ \{b \cdot \text{sgn}\,(\Delta C)\}, & \text{otherwise.} \end{cases}$$

Finally, consider the case when $2\theta_M - \theta_C < 0$. Then, $\Pi(q_1, q_2, r; \boldsymbol{\theta})$ is strictly convex in r, and is maximized at either $r = -b$ or $r = +b$ or both. Again, the analysis mirrors the argument in the proof of Lemma 3, and yields

$$\mathcal{R}(\boldsymbol{\theta}) = \begin{cases} \{\pm b\}, & \text{if} \left|\frac{\Delta C}{2\beta}\right| \leq b, \\ \{b \cdot \text{sgn}\,(\Delta C)\}, & \text{otherwise.} \end{cases}$$

The rest follows from combining the values of $\mathcal{R}(\boldsymbol{\theta})$ under different cases. □