On Convex Relaxation-Based Distribution Locational Marginal Prices

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***Abstract*—We investigate and analyze properties of convex relaxation based prices to compensate distributed energy resources (DERs) connected to a radial distribution grid. Our main contribution is the identification of sufficient conditions under which this distribution locational marginal pricing (DLMP) mechanism supports an efficient market equilibrium and is revenue adequate. We illustrate our analysis through examples.**

***Index Terms*—convex relaxation, locational marginal prices, revenue adequacy, efficient market equilibrium**

I. Introduction

The deepening penetration of distributed energy resources (DERs) coupled with the need to harness demand flexibility of end-use consumers has led to a rapidly increasing interest in defining appropriate price signals for distribution networks [1]–[5]. The proposals have tried to emulate the experience from wholesale electricity markets and locational marginal prices (LMPs). These prices have been commonly referred to as distribution LMPs or DLMPs. Ideally, such prices will encode grid requirements that DERs will respond to.

The core theoretical framework for spot pricing of electricity [6] is ideally suited to a lossless linearized network model, often utilizing the so-called DC approximations[1]. Prices defined using market clearing problems with linearized power flow models exhibit a number of desirable qualities. These properties are typically a consequence of the convexity of the market clearing problem. Such linearizations, however, often ignore considerations of losses, voltage magnitudes and reactive power in the network–considerations that cannot be ignored in distribution grids. Specifically, distribution grids have relatively high resistance to reactance ratios and reactive power injections play an important role in maintaining voltage magnitudes within specified limits. Thus, a direct extension of LMPs with DC approximations of power flow equations to the distribution grids is not appropriate.

In order to take into account the necessary characteristics of distribution networks, we derive our prices for real and reactive power from a second-order cone programming (SOCP) based convex relaxation of the dispatch problem with nonlinear AC power flow equations. Popularized by [8], [9], these relaxations seek to optimize grid assets over a convex set that contains the feasible set described by the power flow equations. The SOCP based convex relaxation with branch flow model has been extensively analyzed in [10]–[12] and is particularly suitable for radial distribution networks. Under certain conditions, these relaxations often yield an optimal solution that satisfies the power flow equations, e.g., see [12], [13]. Authors of [2], [14] have utilized duality theory of this SOCP relaxation to define DLMPs. However, they do not accompany their proposals with an analysis that justifies their design from an economic standpoint—the precise focus of our current effort.

In this paper, we restrict our attention to the mathematical foundations of DLMPs and sidestep issues surrounding the adoption of such pricing schemes to harness DERs. See [15] for insightful discussions on the same. We align with the view in [4] to consider a retail market operated by an independent distribution system operator (DSO) responsible for dispatch and pricing of resources in the distribution network.

In Section II, we describe the branch flow model for radial distribution networks that we utilize to formulate the dispatch and the pricing problems in Section III. In Section IV, we identify conditions under which the DLMPs—derived from the SOCP relaxation of the dispatch problem—support an efficient market equilibrium and are revenue adequate. These properties provide the economic rationale behind adopting DLMPs derived from SOCP-based relaxations in a retail market environment. We illustrate our theoretical results via illustrative examples in Section V, and conclude the paper in Section VI.

II. Modeling the Radial Distribution Network

Distribution grids are often multi-phase unbalanced networks with components such as capacitor banks and tap-changing transformers that play a vital role in maintaining voltage magnitudes within specified limits. In this work, we ignore these special characteristics and model the distribution grid as a single-phase equivalent of a three-phase network, where controllable and uncontrollable assets operated by asset-owners connect at the various buses of the network.

Throughout, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. For $y \in \mathbb{C}$, denote its real and imaginary parts by $Re(y)$ and $Im(y)$, respectively, and $\boldsymbol{i} := \sqrt{-1}$. Vectors and matrices are distinguished with boldfaced letters.

Consider a radial electric distribution network on n buses, the collection of which is defined by $\mathbb{N}$. By radial, we mean that the network does not contain any cycles. Represent the network by a directed graph with $\mathbb{E}$ as the collection of m directed edges. For k and ℓ in $\mathbb{N}$, the edge $k \to \ell \in \mathbb{E}$ represents a line joining k and ℓ. The directions of the edges are chosen arbitrarily.

Let $V_k \in \mathbb{C}$ be the voltage phasor at each bus $k \in \mathbb{N}$. For each $k \to \ell \in \mathbb{E}$, Ohm's law dictates

$$V_k - V_\ell = z_{k\ell} I_{k\ell}, \tag{1}$$

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[1]See [7] for the perils of ad hoc measures to include losses.

where $z_{k\ell} := r_{k\ell} + \boldsymbol{i}x_{k\ell}$ is the complex impedance of the line and $I_{k\ell}$ is the current phasor from bus k to bus ℓ. Then, the sending-end apparent power flow on $k \to \ell$ is given by

$$S_{k\ell} := P_{k\ell} + \boldsymbol{i}Q_{k\ell} = V_k I_{k\ell}^{\mathsf{H}}, \tag{2}$$

where we use the notation y^{H} to denote the complex conjugate of y. The apparent power received at bus ℓ from bus k is then given by

$$V_\ell I_{k\ell}^{\mathsf{H}} = (V_\ell - V_k) I_{k\ell}^{\mathsf{H}} + V_k I_{k\ell}^{\mathsf{H}} = -z_{k\ell} J_{k\ell} + S_{k\ell}, \tag{3}$$

where $J_{k\ell} := |I_{k\ell}|^2$ denotes the squared current magnitude on line $k \to \ell$. Enforcing power balance at each bus utilizing (2) and (3), we get[2]

$$\begin{aligned} p_k^G - p_k^D &= \sum_{\ell' : k \to \ell'} P_{k\ell'} - \sum_{\ell' : \ell' \to k} \left(P_{\ell' k} - r_{\ell' k} J_{\ell' k} \right), \\ q_k^G - q_k^D &= \sum_{\ell' : k \to \ell'} Q_{k\ell'} - \sum_{\ell' : \ell' \to k} \left(Q_{\ell' k} - x_{k\ell'} J_{\ell' k} \right). \end{aligned} \tag{4}$$

Here we assume that each bus has a controllable generation resource injecting an apparent power of $p_k^G + \boldsymbol{i}q_k^G$ at bus k and an uncontrollable demand drawing $p_k^D + \boldsymbol{i}q_k^D$ at bus k. The demands are assumed known and the generation resources can produce power within known capacity limits, satisfying

$$\underline{p}_k^G \le p_k^G \le \overline{p}_k^G, \qquad \underline{q}_k^G \le q_k^G \le \overline{q}_k^G. \tag{5}$$

An uncontrollable generation resource is modeled as a negative demand in our formulation and a controllable load as negative generation. Associate with the injection of $p_k^G + \boldsymbol{i}q_k^G$ the cost $c_k(p_k^G, q_k^G)$ that is linear in its arguments. Such costs can encode both production costs and disutility of deferred demand. In a retail market environment, such costs will be inferred from supply offers and demand bids. Our results largely continue to hold with general convex costs.

Thermal considerations dictate an upper bound on the amount of current flowing over a line. We include these constraints as limits on both sending-end and receiving-end real powers on $k \to \ell$ in[3]

$$P_{k\ell} \le f_{k\ell}, \qquad r_{k\ell} J_{k\ell} - P_{k\ell} \le f_{k\ell}, \tag{6}$$

where $f_{k\ell} > 0$ denotes the line capacity. Let $w_k := |V_k|^2$ be the squared voltage magnitude at bus k that is constrained as

$$\underline{v}_k^2 \le w_k \le \overline{v}_k^2 \tag{7}$$

with known limits $\underline{v}_k^2$, $\overline{v}_k^2$. From (2), we obtain

$$P_{k\ell}^2 + Q_{k\ell}^2 = |S_{k\ell}|^2 = |V_k|^2 \, |I_{k\ell}|^2 = w_k J_{k\ell} \tag{8}$$

and the definition of w's yield

$$w_k - w_\ell = V_k V_k^{\mathsf{H}} - V_\ell V_\ell^{\mathsf{H}} = V_k z_{k\ell}^{\mathsf{H}} I_{k\ell}^{\mathsf{H}} + z_{k\ell} I_{k\ell} V_\ell^{\mathsf{H}}.$$

Leveraging (2) and (3) in the above equation becomes

$$\begin{aligned} w_k - w_\ell &= z_{k\ell}^{\mathsf{H}} S_{k\ell} + z_{k\ell} \left(S_{k\ell} - z_{k\ell} J_{k\ell} \right)^{\mathsf{H}} \\ &= 2Re\left(z_{k\ell}^{\mathsf{H}} S_{k\ell} \right) - |z_{k\ell}|^2 J_{k\ell} \\ &= 2(P_{k\ell} r_{k\ell} + Q_{k\ell} x_{k\ell}) - (r_{k\ell}^2 + x_{k\ell}^2) J_{k\ell}. \end{aligned} \tag{9}$$

[2]We ignore shunt admittances and associated currents for simplicity.
[3]See [2], [12] for alternate definitions of line capacity limits.

The branch flow model represents Kirchhoff's laws in terms of $\boldsymbol{w}$ and $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{J}$, where the vectors collect the corresponding variables over $\mathbb{N}$ and $\mathbb{E}$. If $\boldsymbol{w}, \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{J}$ satisfy (8), (9), then one can recover voltage phasors $\boldsymbol{V} \in \mathbb{C}^n$ that satisfy (1), (2). See [11] for details.

III. The Market Clearing Procedure

With the branch flow model of the power flow equations written in terms of $\boldsymbol{w}, \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{J}$, we now define the dispatch and the pricing problem that a DSO solves.

A. The Dispatch Problem

The DSO solves the following dispatch problem that seeks to minimize (real and reactive) power procurement costs from controllable assets to meet the needs of the uncontrollable assets over the distribution grid.

$$\text{minimize} \quad \sum_{k=1}^{n} c_k(p_k^G, q_k^G),$$

subject to

$$p_k^G - p_k^D = \sum_{\ell' : k \to \ell'} P_{k\ell'} - \sum_{\ell' : \ell' \to k} \left(P_{\ell' k} - r_{\ell' k} J_{\ell' k} \right), \tag{10a}$$

$$q_k^G - q_k^D = \sum_{\ell' : k \to \ell'} Q_{k\ell'} - \sum_{\ell' : \ell' \to k} \left(Q_{\ell' k} - x_{k\ell'} J_{\ell' k} \right), \tag{10b}$$

$$P_{k\ell} \le f_{k\ell}, \; r_{k\ell} J_{k\ell} - P_{k\ell} \le f_{k\ell}, \tag{10c}$$

$$\underline{p}_k^G \le p_k^G \le \overline{p}_k^G, \; \underline{q}_k^G \le q_k^G \le \overline{q}_k^G, \tag{10d}$$

$$\underline{v}_k^2 \le w_k \le \overline{v}_k^2, \tag{10e}$$

$$w_\ell = w_k - 2(P_{k\ell} r_{k\ell} + Q_{k\ell} x_{k\ell}) + (r_{k\ell}^2 + x_{k\ell}^2) J_{k\ell}, \tag{10f}$$

$$P_{k\ell}^2 + Q_{k\ell}^2 = J_{k\ell} w_k \tag{10g}$$

for $k \in \mathbb{N}, \; k \to \ell \in \mathbb{E}$

over the variables $\boldsymbol{p}^G, \boldsymbol{q}^G, \boldsymbol{w}, \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{J}$. Problem (10) is nonconvex, owing to the quadratic equality constraint in (10g). As will be clear in Section III-B, the pricing problem is a convex relaxation of (10).

B. DLMPs from the Pricing Problem

We relax the nonconvex quadratic equality in the dispatch problem to arrive at the following pricing problem. In what follows, we derive the DLMPs from the optimal Lagrange multipliers of the pricing problem.

$$\begin{aligned} &\text{minimize} && \sum_{k=1}^{n} c_k(p_k^G, q_k^G), \\ &\text{subject to} && P_{k\ell}^2 + Q_{k\ell}^2 \le J_{k\ell} w_k, \text{(10a)} - \text{(10f)} \\ &&& \text{for } k \in \mathbb{N}, \; k \to \ell \in \mathbb{E} \end{aligned} \tag{11}$$

over the variables $\boldsymbol{p}^G, \boldsymbol{q}^G, \boldsymbol{w}, \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{J}$. The inequality constraint in (11) is a relaxation of (10g) in the dispatch problem. Furthermore, it is a second-order cone constraint. The cost and the rest of the constraints being linear in the optimization variables, the pricing problem in (11) can be solved as a second-order cone program (SOCP).

Associate Lagrange multipliers λ_k^p and λ_k^q with the real and reactive power balance constraints (10a)-(10b), respectively for the pricing problem.

Definition 1 (Distribution LMPs)**.** *The relaxation-based DLMPs for real and reactive powers at each bus $k \in \mathbb{N}$ are defined as the optimal Lagrange multipliers $\lambda_k^{p,*}$ and $\lambda_k^{q,*}$ from the SOCP-based pricing problem* (11)*.*

We emphasize that our dispatch is derived from the non-convex dispatch problem in (10), while the electricity prices are obtained from its SOCP relaxation in (11). These prices reflect losses, congestion, and account for reactive power flow in the network as these considerations are explicit in the pricing problem. DLMPs $\lambda_k^{p,*}$ and $\lambda_k^{q,*}$ retain an economic interpretation analogous to transmission LMPs–they represent the short-run marginal cost to the system to supply an additional unit of real and reactive power demand at bus k. A more detailed discussion on the economic interpretation of such DLPMs is given in [2].

If the dispatch decision of the controllable asset is given by $p_k^{G,*}, q_k^{G,*}$, then the DSO pays $\lambda_k^{p,*} p_k^{G,*} + \lambda_k^{q,*} q_k^{G,*}$ to the asset owner-operator. The uncontrollable asset at bus k pays $\lambda_k^{p,*} p_k^D + \lambda_k^{q,*} q_k^D$ to the DSO.

IV. Properties of Relaxation-Based DLMPs

We now investigate the properties entrenched in relaxation-based DLMPs. We begin by describing desirable qualities of prices that we seek to establish.

Definition 2 (Efficient Market Equilibrium)**.** *The prescribed dispatch $(\boldsymbol{p}^G, \boldsymbol{q}^G)$ and prices $(\boldsymbol{\lambda}^p, \boldsymbol{\lambda}^q)$ constitute an efficient market equilibrium, if they satisfy the following conditions.*

- *Individual rationality for all controllable assets: Given the prices λ_k^p, λ_k^q, the controllable asset at each bus $k \in \mathbb{N}$ will produce $p_k^G + \boldsymbol{i} q_k^G$ in an effort to maximize its own profit, i.e., (p_k^G, q_k^G) is an optimizer of*

$$\begin{aligned} &\underset{\tilde{p}^G, \tilde{q}^G}{\text{maximize}} && \lambda_k^p \tilde{p}^G + \lambda_k^q \tilde{q}^G - c_k(\tilde{p}^G, \tilde{q}^G), \\ &\text{subject to} && \underline{p}_k^G \le \tilde{p}^G \le \overline{p}_k^G, \ \underline{q}_k^G \le \tilde{q}_k^G \le \overline{q}_k^G. \end{aligned} \tag{12}$$

- *Market clearing condition: The dispatch meets the power demands $\boldsymbol{p}^D + \boldsymbol{i}\boldsymbol{q}^D$ over the network and induce feasible power flows, i.e., there exists $\boldsymbol{w}, \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{J}$ such that $(\boldsymbol{p}^G, \boldsymbol{q}^G, \boldsymbol{w}, \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{J})$ satisfy* (10a) *-* (10g)*.*
- *Efficiency of dispatch: There exists $\boldsymbol{w}, \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{J}$ such that $(\boldsymbol{p}^G, \boldsymbol{q}^G, \boldsymbol{w}, \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{J})$ optimizes* (10)*.*

Individual rationality ensures that a controllable asset has no incentive to deviate from the DSO's prescribed dispatch, given the prices. Market clearing condition ensures that the dispatch meets the demand requirements over the network and an efficient dispatch ensures that it indeed optimizes the aggregate power procurement costs.

Definition 3 (Revenue Adequacy)**.** *The prescribed dispatch $(\boldsymbol{p}^G, \boldsymbol{q}^G)$ and prices $(\boldsymbol{\lambda}^p, \boldsymbol{\lambda}^q)$ define a revenue adequate market mechanism if the merchandizing surplus given by*

$$MS := \sum_{k=1}^{n} \left[\lambda_k^p \left(p_k^D - p_k^G \right) + \lambda_k^q \left(q_k^D - q_k^G \right) \right] \tag{13}$$

is nonnegative.

Nonnegativity of MS implies that the DSO remains solvent after settling payments with market participants.

A. Main Result

We now present our result that identifies sufficient conditions under which our market mechanism with relaxation-based DLMPs exhibits desirable properties mentioned above. See Section IV-B for its proof.

Theorem 1. *Suppose the dispatch problem* (10) *is strictly feasible. Consider a dispatch $(\boldsymbol{p}^{G,*}, \boldsymbol{q}^{G,*})$ computed from* (10) *and DLMPs $\boldsymbol{\lambda}^{p,*}, \boldsymbol{\lambda}^{q,*}$ computed from* (11)*. If the inequality in* (11) *is tight at an optimum, then this dispatch and DLMPs support an efficient market equilibrium. Moreover, if the lower bound on voltage magnitudes is non-binding at every bus, then they define a revenue adequate market mechanism.*

Theorem 1 establishes the properties of relaxation-based DLMPs under the premise that the SOCP relaxation is exact. In other words, an optimal solution of the convex relaxation of the nonconvex dispatch problem produces an optimal dispatch. There are several sufficient conditions under which this convex relaxation is exact, e.g., see [10], [12], [13]. Even when such conditions do not hold, relaxations over radial distribution networks are often exact.

We establish revenue adequacy under an additional condition on the lower bounds on voltage magnitudes. Our numerical experiments in Section V reveal that this extra condition is sufficient but not necessary for MS ≥ 0. We expect from the long literature on volt-VAR control problems that adequate reactive power support will likely lead to voltage magnitudes higher than their lower limits.

B. Proof of Theorem 1

For (11) for all $k \in \mathbb{N}$ and all $k \to \ell \in \mathbb{E}$, associate the Lagrange multipliers λ_k^p and λ_k^q with (10a) and (10b), respectively. Similarly, associate $\alpha_{k\ell}$ and $\alpha'_{k\ell}$ with the constraints on real power flows on the lines in (10c). Assign $\overline{\mu}_k^p, \underline{\mu}_k^p$ to the upper and lower limits on the real power supply in (10d), and $\overline{\mu}_k^q, \underline{\mu}_k^q$ to the respective limits on the reactive power supply in (10d). Define $\overline{\mu}_k^w, \underline{\mu}_k^w$ as the multipliers for the upper and lower bounds on squared voltage magnitudes in (10e), and $\nu_{k\ell}$ for (10f). Finally, set $\sigma_{k\ell}$ as the multiplier for the second-order cone constraint in (11).

Problem in (11) is convex. With Slater's condition, the Karush-Kuhn-Tucker (KKT) conditions in Figure 1 are necessary and sufficient for optimality.

1) Proof of Efficient Market Equilibrium.: If the inequality in (11) binds at an optimum, then the same solution is feasible and optimal in (10). Then, the dispatch is efficient and satisfies the market clearing condition. To show individual rationality, notice that problem (12) is convex and satisfies weak Slater's condition. Therefore, KKT conditions are both necessary and sufficient for optimality. Associate Lagrange multipliers μ_U^p, μ_L^p to the upper and lower bounds on real power generation in (12) and μ_U^q, μ_L^q to that on reactive power generation. KKT conditions then dictate that $\tilde{p}^{G,*}, \tilde{q}^{G,*}$

- Primal feasibility conditions: (10a) - (10f), $P_{k\ell}^2 + Q_{k\ell}^2 \leq J_{k\ell} w_k$ for $k \in \mathbb{N},\ k \to \ell \in \mathbb{E}$.
- Dual feasibility conditions: $\alpha_{k\ell}^*, \alpha_{k\ell'}^*, \overline{\mu}_k^{p,*}, \underline{\mu}_k^{p,*}, \overline{\mu}_k^{q,*}, \underline{\mu}_k^{q,*}, \underline{\mu}_k^{w,*}, \overline{\mu}_k^{w,*}, \sigma_{k\ell}^* \geq 0$.
- Gradient conditions: For $k \in \mathbb{N},\ k \to \ell \in \mathbb{E}$,

$$\lambda_k^{p,*} - \lambda_\ell^{p,*} + \alpha_{k\ell}^* - {\alpha'_{k\ell}}^* - 2\nu_{k\ell}^* r_{k\ell} + 2P_{k\ell}^* \sigma_{k\ell}^* = 0, \tag{14a}$$

$$\lambda_k^{q,*} - \lambda_\ell^{q,*} - 2\nu_{k\ell}^* x_{k\ell} + 2Q_{k\ell}^* \sigma_{k\ell}^* = 0, \tag{14b}$$

$$\lambda_\ell^{p,*} r_{k\ell} + \lambda_\ell^{q,*} x_{k\ell} + {\alpha'_{k\ell}}^* r_{k\ell} + \nu_{k\ell}^* (r_{k\ell}^2 + x_{k\ell}^2) - \sigma_{k\ell}^* w_k^* = 0, \tag{14c}$$

$$\overline{\mu}_k^{w,*} - \underline{\mu}_k^{w,*} + \sum_{\ell: k\to\ell} \nu_{k\ell}^* - \sum_{\ell: \ell\to k} \nu_{\ell k}^* - \sum_{\ell: k\to\ell} \sigma_{k\ell}^* J_{k\ell}^* = 0, \tag{14d}$$

$$\nabla_{p_k^G} c_k \left(p_k^{G,*}, q_k^{G,*}\right) - \lambda_k^{p,*} + \overline{\mu}_k^{p,*} - \underline{\mu}_k^{p,*} = \nabla_{q_k^G} c_k \left(p_k^{G,*}, q_k^{G,*}\right) - \lambda_k^{q,*} + \overline{\mu}_k^{q,*} - \underline{\mu}_k^{q,*} = 0. \tag{14e}$$

- Complementary slackness conditions: For $k \in \mathbb{N},\ k \to \ell \in \mathbb{E}$,

$$\alpha_{k\ell}^*(P_{k\ell}^* - f_{k\ell}) = {\alpha'_{k\ell}}^*(r_{k\ell} J_{k\ell}^* - P_{k\ell}^* - f_{k\ell}) = \sigma_{k\ell}^* \left([P_{k\ell}^*]^2 + [Q_{k\ell}^*]^2 - J_{k\ell}^* w_k^* \right) = 0, \tag{15a}$$

$$\underline{\mu}_k^{p,*} \left(p_k^{G,*} - \underline{p}_k^G\right) = \overline{\mu}_k^{p,*} \left(p_k^{G,*} - \overline{p}_k^G\right) = \underline{\mu}_k^{q,*} \left(q_k^{G,*} - \underline{q}_k^G\right) = \overline{\mu}_k^{q,*} \left(q_k^{G,*} - \overline{q}_k^G\right) = 0, \tag{15b}$$

$$\underline{\mu}_k^{w,*} \left(w_k^* - \underline{v}_k^2\right) = \overline{\mu}_k^{w,*} \left(w_k^* - \overline{v}_k^2\right) = 0. \tag{15c}$$

Fig. 1: The Karush-Kuhn-Tucker (KKT) optimality conditions for (11).

and $\mu_U^{p,*}, \mu_L^{p,*}$, $\mu_U^{q,*}, \mu_L^{q,*} \geq 0$ identify a primal-dual optimal solution of (12), if they satisfy

$$\begin{aligned}
\nabla_{p_k^G} c_k \left(\tilde{p}^{G,*}, \tilde{q}^{G,*}\right) - \lambda_k^{p,*} + \mu_U^{p,*} - \mu_L^{p,*} = 0,\\
\nabla_{q_k^G} c_k \left(\tilde{p}^{G,*}, \tilde{q}^{G,*}\right) - \lambda_k^{q,*} + \mu_U^{q,*} - \mu_L^{q,*} = 0,\\
\mu_L^{p,*} \left(\tilde{p}^{G,*} - \underline{p}_k^G\right) = \mu_U^{p,*} \left(\tilde{p}^{G,*} - \overline{p}_k^G\right) = 0,\\
\mu_L^{q,*} \left(\tilde{q}^{G,*} - \underline{q}_k^G\right) = \mu_U^{q,*} \left(\tilde{q}^{G,*} - \overline{q}_k^G\right) = 0.
\end{aligned}$$

The KKT conditions of (11) in Figure 1 imply these with

$$\tilde{p}^{G,*} = p_k^{G,*}, \tilde{q}^{G,*} = q_k^{G,*}, \mu_U^{p,*} = \overline{\mu}^{p,*}, \mu_L^{p,*} = \underline{\mu}^{p,*},$$
$$\mu_U^{q,*} = \overline{\mu}^{q,*}, \mu_L^{q,*} = \underline{\mu}^{q,*}.$$

Therefore, $(p_k^{G,*}, q_k^{G,*})$ optimizes (12), completing the proof.

2) Proof of Revenue Adequacy.: Consider a primal-dual optimal solution of (11). Drop '*' from the notation for convenience. From (13), we infer

$$\begin{aligned}
\text{MS} &= \sum_{k=1}^n \lambda_k^p \left(p_k^D - p_k^G\right) + \lambda_k^q (q_k^D - q_k^G)\\
&= \sum_{k=1}^n -\lambda_k^p \left[\sum_{\ell: k\to\ell} P_{k\ell} - \sum_{\ell: \ell\to k} \left(P_{\ell k} - r_{\ell k} J_{\ell k}\right) \right]\\
&\quad + \sum_{k=1}^n -\lambda_k^q \left[\sum_{\ell: k\to\ell} Q_{k\ell} - \sum_{\ell: \ell\to k} \left(Q_{\ell k} - x_{k\ell} J_{\ell k}\right) \right]\\
&= \sum_{k=1}^n \sum_{\ell: k\to\ell} \left(T_{k\ell}^1 + T_{k\ell}^2 - T_{k\ell}^3\right),
\end{aligned} \tag{16}$$

where

$$T_{k\ell}^1 := P_{k\ell} \left(\lambda_\ell^p - \lambda_k^p\right), \quad T_{k\ell}^2 := Q_{k\ell} \left(\lambda_\ell^q - \lambda_k^q\right),$$
$$T_{k\ell}^3 := J_{k\ell} \left(\lambda_\ell^p r_{k\ell} + \lambda_\ell^q x_{k\ell}\right).$$

Simplifying $T_{k\ell}^1$ using (14a) and $T_{k\ell}^2$ using (14b), we get

$$\begin{aligned}
T_{k\ell}^1 + T_{k\ell}^2 &= P_{k\ell} \left(\alpha_{k\ell} - \alpha'_{k\ell}\right) - 2\nu_{k\ell} P_{k\ell} r_{k\ell} + 2P_{k\ell}^2 \sigma_{k\ell}\\
&\quad - 2\nu_{k\ell} Q_{k\ell} x_{k\ell} + 2Q_{k\ell}^2 \sigma_{k\ell}\\
&= P_{k\ell} \left(\alpha_{k\ell} - \alpha'_{k\ell}\right) - 2\nu_{k\ell} \left(P_{k\ell} r_{k\ell} + Q_{k\ell} x_{k\ell}\right)\\
&\quad + 2\sigma_{k\ell} w_k J_{k\ell}.
\end{aligned} \tag{17}$$

We have utilized (10g) in the derivation of the last line. Furthermore, utilizing (14c), we obtain

$$\begin{aligned}
T_{k\ell}^3 &= -\alpha'_{k\ell} J_{k\ell} r_{k\ell} + \sigma_{k\ell} w_k J_{k\ell} - \nu_{k\ell} J_{k\ell} \left(r_{k\ell}^2 + x_{k\ell}^2\right)\\
&= -\alpha'_{k\ell} J_{k\ell} r_{k\ell} + \sigma_{k\ell} w_k J_{k\ell}\\
&\quad - \nu_{k\ell} \left[w_\ell - w_k + 2\left(P_{k\ell} r_{k\ell} + Q_{k\ell} x_{k\ell}\right)\right],
\end{aligned} \tag{18}$$

where the last line follows from (10f). Combination of (17) and (18) with elementary algebra yield

$$\begin{aligned}
T_{k\ell}^1 + T_{k\ell}^2 - T_{k\ell}^3 &= \alpha_{k\ell} P_{k\ell} - \alpha'_{k\ell} P_{k\ell} + \alpha'_{k\ell} J_{k\ell} r_{k\ell}\\
&\quad + \nu_{k\ell}(w_\ell - w_k) + \sigma_{k\ell} w_k J_{k\ell}\\
&= \alpha_{k\ell} f_{k\ell} + \alpha'_{k\ell} f_{k\ell}\\
&\quad + \nu_{k\ell}(w_\ell - w_k) + \sigma_{k\ell} w_k J_{k\ell}\\
&\geq \nu_{k\ell}(w_\ell - w_k) + \sigma_{k\ell} w_k J_{k\ell},
\end{aligned}$$

where the equality utilizes the complementary slackness conditions in (15a) from Figure 1, and the inequality follows from the nonnegativity of $\boldsymbol{f}, \boldsymbol{\alpha}, \boldsymbol{\alpha}'$. Plugging the above relation in (16), we deduce

$$\text{MS} \geq \sum_{k=1}^n \sum_{\ell: k\to\ell} \left[\nu_{k\ell}(w_\ell - w_k) + \sigma_{k\ell} w_k J_{k\ell}\right]. \tag{19}$$

Multiply the gradient condition in (14d) with w_k and sum the resulting equation over all $k \in \mathbb{N}$ to get

$$\sum_{k=1}^n \sum_{\ell: k\to\ell} \sigma_{k\ell} w_k J_{k\ell} + \nu_{k\ell} \left(w_\ell - w_k\right) = \sum_{k=1}^n \overline{\mu}_k^w w_k \geq 0,$$

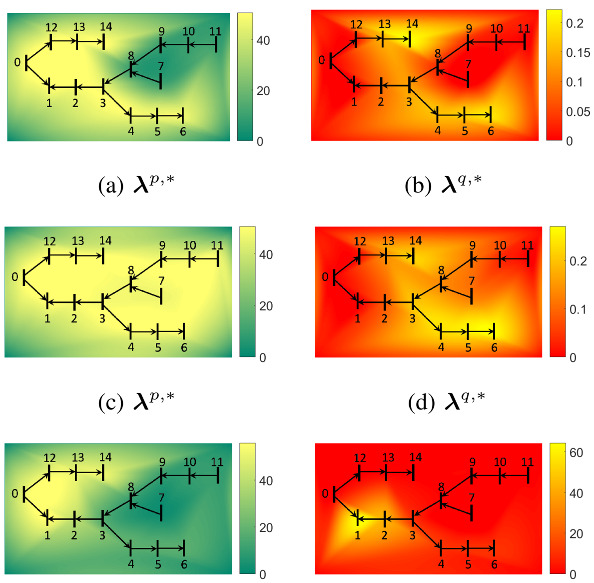
implying MS ≥ 0. We have utilized the fact that non-binding lower limits on voltage magnitudes yield $\underline{\mu}_k^w = 0$ from (15b). Nonnegativity of $\overline{\boldsymbol{\mu}}^w$, $\boldsymbol{w}$ implies the rest.

V. Numerical Experiments

The pricing problem (11) for all experiments was solved in CVX in MATLAB. Power is reported in per units.

#	f_{12}	r_{12}	x_{12}	k	p_k^D	q_k^D	$\overline{p}_k^G$	$\overline{q}_k^G$	$\underline{v}_k^2$	$\overline{v}_k^2$	c_k	w_k^*	MS
1	0.5	.10	.10	1	1.6	0	2.0	2.0	.81	1.20	10	1.20	+0.27
				2	2.0	.2	2.0	2.0	.81	1.20	20	1.12	
2	1.0	.10	.10	1	1.0	.5	2.0	1.0	.95	1.10	8	1.10	+0.71
				2	2.8	.5	2.0	1.5	.95	1.10	5	0.95	
3	1.0	.01	.01	1	0.8	.5	2.0	0.9	.86	0.95	10	0.95	-0.10
				2	0	0	1.2	0.2	.97	1.10	10	0.97	

TABLE I: Parameter choices and outcomes of the pricing problem for our experiments on the 2-bus network.



(a) $\boldsymbol{\lambda}^{p,*}$ (b) $\boldsymbol{\lambda}^{q,*}$

(c) $\boldsymbol{\lambda}^{p,*}$ (d) $\boldsymbol{\lambda}^{q,*}$

(e) $\boldsymbol{\lambda}^{p,*}$ (f) $\boldsymbol{\lambda}^{q,*}$

Fig. 2: Plots (a), (b) show heatmaps of DLMPs on the 15-bus radial network adopted from [2]. Plots (c), (d) are derived with $p_{11}^D = 0.350$, and (e), (f) with $\overline{v}_i^2 = 1.05, i = 0, \ldots, 10,\ \underline{v}_1^2 = 1$.

A. On a 2-bus network example

We conduct three experiments on a 2-bus network with linear dispatch cost of the form $c_1 p_1^G + c_2 p_2^G$. Parameters for and outcomes of our experiments are given in Table I. We set the lower limits $\underline{p}_k^G = 0$, $\underline{q}_k^G = 0$ for all buses in all experiments, except for $\underline{q}_2^G = 0.01$ in the third experiment. All relaxations were verified to be exact. In the first experiment, the voltage magnitude at each bus is strictly greater than the lower limit and we obtain MS ≥ 0, as Theorem 1 dictates. In the second experiment, voltage magnitude at bus 2 equals its lower limit, but we still obtain MS ≥ 0, revealing that Theorem 1 identifies a sufficient but not a necessary condition for nonnegative MS. Our third experiment violates the sufficient condition and yields a negative MS.

B. On a 15-bus network example

Figure 2 portrays DLMPs on a 15-bus radial network from [2] with the modification $p_{11}^D = 0.250$ and $q_{11}^D = 0.073$. Figures 2c, 2d reveal that increasing power demands at bus 11 increases real power prices around bus 11, illustrating the locational nature of these prices. Figures 2e,2f demonstrate that voltage limits significantly affect reactive power prices. Our experiments with various parameters always yielded MS ≥ 0.

VI. Conclusions and Future Directions

In this paper, we identify sufficient conditions under which DLMPs for real and reactive power derived from an SOCP-based market clearing problem support an efficient market equilibrium and is revenue adequate. We illustrate our results through numerical examples. Our proof techniqu do not easily extend to cases where the relaxation is not exact – a case that requires further study. Extension of our work to consider multi-phase unbalanced distribution grid model with capacitor banks and tap-changing transformers is another interesting direction for future research.

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